Journal of Algebra 480 (2017) 368–384

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A new involution for quantum loop algebras

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A R T I C L E   I N F O

Article history:
Received 4 November 2016
Available online 20 March 2017
Communicated by Volodymyr Mazorchuk

MSC:
16T20
81R50

Keywords:
Quantum affine algebras
Quantum groups

A B S T R A C T

In this article, we introduce a completion $\widehat{\mathbf{U}}_v^+(\mathcal{L}\mathfrak{g})$ of the positive half of the quantum affinization $\mathbf{U}_v^+(\mathcal{L}\mathfrak{g})$ of a symmetrizable Kac–Moody algebra $\mathfrak{g}$. On $\widehat{\mathbf{U}}_v^+(\mathcal{L}(\mathfrak{g}))$, we define a new "bar-involution" and construct the analogue Kashiwara's operators. We conjecture that the resulting pair $(\widehat{\mathcal{L}}, \widehat{\mathcal{B}})$ is a crystal basis which provides the existence of the "canonical basis" on the (completion of the) of the positive half of the quantum affinization.

1. Introduction

One of the main outcomes of the theory of quantum groups is the discovery, by Kashiwara [7] and Lusztig [8], of the canonical bases in quantized enveloping algebras with certain favourable properties: positivity of structure constants, compatibility with all highest weight integrable representations, etc. These canonical bases have been proven to be powerful tool in the study of the representation theory of quantum Kac–Moody algebras, encoding character formulas and decomposition numbers.

The construction of the canonical basis $\mathbf{B}$ of a quantized enveloping algebra $\mathbf{U}_v^+(\mathfrak{g})$ in [8] is based on Ringel's discovery in 90s that $\mathbf{U}_v^+(\mathfrak{g})$ can be realized in the Hall algebra

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http://dx.doi.org/10.1016/j.jalgebra.2017.03.012
0021-8693/

of the category of representations of a quiver Q with underlying graph is the Dynkin diagram of $\mathfrak{g}$ ([11]). The set of isomorphism classes of representations of Q of a given class $\mathbf{d} \in K_0(\mathrm{Rep}\, Q)$ is the set of orbits of a reductive group $G_{\mathbf{d}}$ on a vector space $E_{\mathbf{d}}$ and Lusztig realizes $\mathbf{U}_v^+(\mathfrak{g})$ geometrically as a convolution algebra of semisimple, $G_{\mathbf{d}}$-equivariant constructible sheaves on $E_{\mathbf{d}}$ and obtains the canonical basis as the set of all simple perverse sheaves on this algebra. In [6], Kapranov shows that the Hall algebra of the category of coherent sheaves on a smooth projective line provides a realization of the affinization (cf. [4]) $\mathbf{U}_v^+(\mathcal{L}(\mathfrak{sl}_2))$ of the Drinfeld's positive part of the quantum affine algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_2)$. Schiffmann then constructs in [12] the canonical basis $\widehat{\mathcal{B}}$ of the completion $\widehat{\mathbf{U}}_v^+(\mathcal{L}\mathfrak{sl}_2)$ in a similar fashion and prove the compatibility with some integrable lowest weight representations.

Kashiwara's scheme to construct the canonical basis is quite different from Lusztig's one and his approach makes sense for all symmetrizable Kac–Moody algebra $\mathfrak{g}$. The main ingredients of [7] are certain operators (called Kashiwara's operators) $\widetilde{E}_i, \widetilde{F}_i : \mathbf{U}_v^+(\mathfrak{g}) \to \mathbf{U}_v^+(\mathfrak{g})$ for all $i \in I$ to generate the $\mathcal{A}$-lattice $\mathcal{L}$ of $\mathbf{U}_v^+(\mathfrak{g})$ and the basis $\mathcal{B}$ of $\mathcal{L}/v\mathcal{L}$, where $\mathcal{A}$ is the localization of $\mathbb{Q}[v]$ at $v = 0$. Such a pair $(\mathcal{L}, \mathcal{B})$ stable by Kashiwara's operators is called a crystal basis of $\mathbf{U}_v^+(\mathfrak{g})$. Consider the so-called bar-involution $\varphi : \mathbf{U}_v^+(\mathfrak{g}) \to \mathbf{U}_v^+(\mathfrak{g})$ defined by $v \mapsto v^{-1}, E_i \mapsto E_i$ and let $\mathcal{L}^- = \varphi(\mathcal{L})$. Then there is an isomorphism $\mathbf{U}_v^+(\mathfrak{g}) \cap \mathcal{L} \cap \mathcal{L}^- \simeq \mathcal{L}/v\mathcal{L}$ and the pre-image of $\mathcal{B}$ under the isomorphism is a basis of $\mathbf{U}_v^+(\mathfrak{g})$ which coincides with the canonical basis obtained by Lusztig. Such a triple $(\mathbf{U}_v^+(\mathfrak{g}), \mathcal{L}, \mathcal{L}^-)$ equipped with the above isomorphism is called a *balanced triple* and its existence is equivalent to the existence of the canonical basis.

In this short paper, we develop a purely algebraic approach (under the scheme of Kashiwara in [7]) to Schiffmann's canonical basis $\widehat{\mathcal{B}}$ on $\widehat{\mathbf{U}}_v^+(\mathcal{L}\mathfrak{sl}_2)$ and extend the construction to the (positive part of the) quantum affinization of symmetrizable Kac–Moody algebra: We provide a construction of Kashiwara's operators adapted to the Drinfeld's half part of the quantum affinization $\mathbf{U}_v(\mathcal{L}\mathfrak{g})$ for all symmetrizable Kac–Moody algebra $\mathfrak{g}$ and use them to generate the (conjectural) crystal basis $(\mathcal{L}, \mathcal{B})$. To generalize the concept of the canonical basis, we extend the bar-involution φ induced by Verdier duality in the context of $\widehat{\mathbf{U}}_v^+(\mathcal{L}\mathfrak{sl}_2)$ to the general cases which seems to be unknown before in the rich study of quantum affine algebras (cf. [1] and references therein). Since the image of the involution φ involves infinite sums, we introduce a certain completion $\widehat{\mathbf{U}}_v^+(\mathcal{L}\mathfrak{g})$ of $\mathbf{U}_v^+(\mathcal{L}\mathfrak{g})$ and the resulting (conjectural) canonical basis $\widehat{\mathcal{B}}$ (if it exists) should actually lie in the completion.

Unfortunately the construction of Kashiwara's operators can not apply to the highest l-weight representations (cf. [1]) since there is not known non-degenerate bilinear form on them and therefore we fail to generalize the "Grand Loop" argument to our cases. In [5], Joseph's refinement of Grand Loop argument relies on the representation theory of quantum Weyl algebras which, in our cases, there is an "affinization" of quantum Weyl algebra whose representations seems worth to study. On the other hand, the Grand Loop argument relies heavily on the tensor product of integrable highest weight representations. By considering instead the "fusion product" of highest l-weight modules (cf. [4])