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Invariant forms on irreducible modules of simple algebraic groups

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ABSTRACT

Let G be a simple linear algebraic group over an algebraically closed field K of characteristic $p \geq 0$ and let V be an irreducible rational G-module with highest weight λ. When V is self-dual, a basic question to ask is whether V has a non-degenerate G-invariant alternating bilinear form or a non-degenerate G-invariant quadratic form.

If $p \neq 2$, the answer is well known and easily described in terms of λ. In the case where $p = 2$, we know that if V is self-dual, it always has a non-degenerate G-invariant alternating bilinear form. However, determining when V has a non-degenerate G-invariant quadratic form is a classical problem that still remains open. We solve the problem in the case where G is of classical type and λ is a fundamental highest weight ω_i, and in the case where G is of type A_l and $\lambda = \omega_r + \omega_s$ for $1 \leq r < s \leq l$. We also give a solution in some specific cases when G is of exceptional type.

As an application of our results, we refine Seitz's 1987 description of maximal subgroups of simple algebraic groups of classical type. One consequence of this is the following result. If $X < Y < \mathrm{SL}(V)$ are simple algebraic groups and $V \downarrow X$ is irreducible, then one of the following holds: (1) $V \downarrow Y$ is not self-dual; (2) both or neither of the modules

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$V \downarrow Y$ and $V \downarrow X$ have a non-degenerate invariant quadratic form; (3) $p = 2$, $X = \mathrm{SO}(V)$, and $Y = \mathrm{Sp}(V)$.

1. Introduction

Let V be a finite-dimensional vector space over an algebraically closed field K of characteristic $p \geq 0$.

A fundamental problem in the study of simple linear algebraic groups over K is the determination of maximal closed connected subgroups of simple groups of classical type ($\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$). Seitz [40] has shown that up to a known list of examples, these are given by the images of p-restricted, tensor-indecomposable irreducible rational representations $\varphi : G \to \mathrm{GL}(V)$ of simple algebraic groups G over K.

Then given such an irreducible representation φ, one should still determine which of the groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$ contain $\varphi(G)$. In most cases the answer is known.

- If V is not self-dual, then $\varphi(G)$ is only contained in $\mathrm{SL}(V)$. Furthermore, we know when V is self-dual (see Section 2).
- If $p \neq 2$ and V is self-dual, then $\varphi(G)$ is contained in $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$, but not both [41, Lemma 78, Lemma 79]. Furthermore, we know for which irreducible representations the image is contained in $\mathrm{Sp}(V)$ and for which the image is contained in $\mathrm{SO}(V)$ (see Section 2).
- If $p = 2$ and V is self-dual, then $\varphi(G)$ is contained in $\mathrm{Sp}(V)$ [17, Lemma 1].

Currently what is still missing is a method for determining in characteristic two when exactly $\varphi(G)$ is contained in $\mathrm{SO}(V)$. This problem is the main subject of this paper, and we can state it equivalently as follows.

Problem 1.1. Assume that $p = 2$ and let $L_G(\lambda)$ be an irreducible G-module with highest weight λ. When does $L_G(\lambda)$ have a non-degenerate G-invariant quadratic form?

This is a nontrivial open problem. There is some literature on the subject [47,43,22,23,19], but currently only partial results are known. The main result of this paper is a solution to Problem 1.1 in the following cases:

- G is of classical type (A_l, B_l, C_l or D_l) and λ is a fundamental dominant weight ω_r for some $1 \leq r \leq l$ (Theorem 4.2).
- G is of type A_l and $\lambda = \omega_r + \omega_s$ for $1 \leq r < s \leq l$ (Theorem 5.1).

In the case where G is of exceptional type, we will give some partial results in Section 6. For G of type G_2 and F_4, we are able to give a complete solution (Proposition 6.1,