Journal of Algebra 480 (2017) 487–504

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Quiver generalization of a conjecture of King, Tollu, and Toumazet

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A R T I C L E I N F O

Article history:
Received 1 July 2016
Available online 21 March 2017
Communicated by Shrawan Kumar

Keywords:
Quivers
Quiver representations
Representation theory
Littlewood–Richardson coefficients
Algebraic geometry
Geometric invariant theory
GIT
Moduli spaces

A B S T R A C T

Stretching the parameters of a Littlewood–Richardson coefficient of value 2 by a factor of n results in a coefficient of value $n+1$ [12,9,19]. We give a geometric proof of a generalization for representations of quivers.

1. Introduction

The Littlewood–Richardson coefficients $c^{\nu}_{\lambda,\mu}$ arise in the representation theory of the general linear group. They depend on tuples of nonnegative integers (weights) λ, μ, and ν. An operation called stretching can be performed in which all of the integers in the tuples λ, μ, and ν are multiplied by n. The effect of this on the Littlewood–Richardson coefficient, that is, the function $P(n) := c^{n\nu}_{n\lambda,n\mu}$, has been studied by many. A number of new and existing conjectures on the behavior of P were summarized by King et al. [12].

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http://dx.doi.org/10.1016/j.jalgebra.2017.03.011
0021-8693/

We list some of these below. Assume $|\lambda| + |\mu| = |\nu|$ (which is anyway necessary for $P(1) \neq 0$). Then...

- (Polynomiality Conjecture.) P is a polynomial with rational coefficients.
- (Saturation Conjecture.) If $P(1) = 0$, then $P(n) = 0$ for all $n \geq 1$.
- (Fulton's Conjecture.) If $P(1) = 1$, then $P(n) = 1$ for all $n \geq 1$.
- (KTT Conjecture.) If $P(1) = 2$, then $P(n) = n + 1$ for all $n \geq 1$.

The polynomiality conjecture was proven by Derksen and Weyman [7]. The first (combinatorial) proofs of the saturation and Fulton conjectures are due to Knutson, Tao, and Woodward [11,13]. Subsequent geometric proofs appeared from Belkale [2,3] and others, which allow for an arbitrary number of weights after symmetrizing. The KTT conjecture was proven combinatorially by Ikenmeyer [9] for three weights, and geometrically by the author [19], again symmetrizing and allowing for an arbitrary number of weights.

For α, β dimension vectors of a cycle-free quiver Q with Ringel product 0, the dimensions of the spaces of σ_β-semi-invariant functions $\mathrm{SI}(Q, \alpha)_{\sigma_\beta}$ on $\mathrm{Rep}(Q, \alpha)$ appear to exhibit the same behavior under stretching as the Littlewood–Richardson numbers (see Section 2 for notation and generalities on quiver representations). Thus, one can make the same assertions for the function $\widetilde{P}(n) := \dim \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}}$.

- (Polynomiality.) $\widetilde{P}$ is a polynomial with rational coefficients.
- (Saturation.) If $\widetilde{P}(1) = 0$, then $\widetilde{P}(n) = 0$ for all $n \geq 1$.
- (Fulton.) If $\widetilde{P}(1) = 1$, then $\widetilde{P}(n) = 1$ for all $n \geq 1$.

All of the above were proven by Derksen and Weyman in the papers [7,6,8], respectively, the last of these having been translated from work of Belkale. It is well-known that the results for $\widetilde{P}$ imply those for P, the Littlewood–Richardson numbers coinciding with dimensions of spaces of semi-invariant functions for special choices of Q, α, β (see Section 9 for one approach). The main object of this paper is to establish the corresponding quiver generalization of the KTT Conjecture. That is, we prove:

Theorem 1.1. *Let α, β be dimension vectors of Q, a quiver without oriented cycles, such that $\langle \alpha, \beta \rangle_Q = 0$. If $\dim \mathrm{SI}(Q, \alpha)_{\sigma_\beta} = 2$, then $\dim \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}} = n + 1$ for all positive integers n.*

Our approach proceeds through geometric invariant theory, following similar proofs in [3,19]. Along the way, we prove by dimension counting a result of general interest, Proposition 4.1. It has the flavor of results from Schofield's paper [17], in that it equates $\dim \mathrm{Ext}_Q(V, W)$ with $\dim \mathrm{Ext}_Q(S, W)$, where S is a certain subrepresentation of V.

In the last section, we show how to deduce the main result on Littlewood–Richardson coefficients of the author's paper [19] (restated as Corollary 9.4 here) from Theorem 1.1. Although we have stressed above the important relationship between quivers and rep-