Journal of Algebra 480 (2017) 505–518

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Exact factorizations and extensions of fusion categories

Shlomo Gelaki

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 10 November 2016
Available online 21 March 2017
Communicated by Nicolás Andruskiewitsch

Keywords:
Fusion categories
Module categories
Exact sequences of fusion categories
Exact factorizations of fusion categories

ABSTRACT

We introduce and study the new notion of an *exact factorization* $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ of a fusion category $\mathcal{B}$ into a product of two fusion subcategories $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ of $\mathcal{B}$. This is a categorical generalization of the well known notion of an exact factorization of a finite group into a product of two subgroups. We then relate exact factorizations of fusion categories to exact sequences of fusion categories with respect to an indecomposable module category, which was introduced and studied in [7]. We also apply our results to study extensions of a group-theoretical fusion category by another one, provide some examples, and propose a few natural questions.

1. Introduction

Finite groups with exact factorization are fundamental objects in group theory which naturally show up in many interesting results in the theories of Hopf algebras and tensor categories (see, e.g., [6,11,12,14,16]). Recall that a finite group G admits an exact factorization $G = G_1 G_2$ into a product of two subgroups $G_1, G_2 \subseteq G$ of G if G_1 and G_2 intersect trivially and the order of G is the product of the orders of G_1 and G_2.

E-mail address: gelaki@math.technion.ac.il.

http://dx.doi.org/10.1016/j.jalgebra.2017.02.034
0021-8693/

Equivalently, $G = G_1G_2$ is an exact factorization if every element $g \in G$ can be uniquely written in the form $g = g_1g_2$, where $g_1 \in G_1$ and $g_2 \in G_2$.

Our first goal in this paper is to provide a categorical generalization of the notion of exact factorizations of finite groups. More precisely, we introduce and study the new notion of an *exact factorization* $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ of a fusion category $\mathcal{B}$ into a product of two fusion subcategories $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ of $\mathcal{B}$. We say that $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization if $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$ and $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\mathrm{FPdim}(\mathcal{C})$. Then in Theorem 3.8 we prove that $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ if and only if every simple object of $\mathcal{B}$ can be uniquely expressed in the form $X \otimes Y$, where X, Y are simple objects of $\mathcal{A}, \mathcal{C}$, respectively. For example, exact factorizations $\mathcal{B} = \mathrm{Vec}(G_1) \bullet \mathrm{Vec}(G_2)$ are classified by groups G with exact factorization $G = G_1G_2$ and a cohomology class $\omega \in H^3(G, k^\times)$ which is trivial on G_1 and G_2 (but not necessarily on G).

Recall next that the theory of exact sequences of tensor categories was introduced by A. Bruguières and S. Natale [4,5] as a categorical generalization of the theory of exact sequences of Hopf algebras. In their definition of an exact sequence of tensor categories

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

the category $\mathcal{A}$ is forced to have a tensor functor to Vec (so to be the representation category of a Hopf algebra). Later on in [7] we generalized the definition of [4] further to eliminate this drawback, and in particular to include the example of the Deligne tensor product $\mathcal{B} := \mathcal{A} \boxtimes \mathcal{C}$ for any finite tensor categories $\mathcal{A}$, $\mathcal{C}$. We did so by replacing the category Vec by the category $\mathrm{End}(\mathcal{N})$ of endofunctors of an indecomposable $\mathcal{A}$-module category $\mathcal{N}$, and defined the notion of an *exact sequence*

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \boxtimes \mathrm{End}(\mathcal{N}) \tag{1}$$

with respect to $\mathcal{N}$. We showed that the dual of an exact sequence is again an exact sequence. We also showed that for any exact sequence (1),

$$\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\mathrm{FPdim}(\mathcal{C}),$$

and that this property in fact characterizes exact sequences (provided that ι is injective, F is surjective, and $\mathcal{A} \subseteq \mathrm{Ker}(F)$). Moreover, we showed that if in an exact sequence (1), $\mathcal{A}$ and $\mathcal{C}$ are fusion categories, then so is $\mathcal{B}$.

Our second goal in this paper is to relate exact factorizations of fusion categories with exact sequences. More precisely, in Theorem 4.1 we prove that an exact sequence of fusion categories (1) defines an exact factorization $\mathcal{B}^*_{\mathcal{C}\boxtimes\mathcal{N}} = \mathcal{C} \bullet \mathcal{A}^*_{\mathcal{N}}$, and vice versa, any exact factorization $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ of fusion categories gives rise to an exact sequence (1) with respect to any indecomposable module category $\mathcal{N}$ over $\mathcal{A}$.

The structure of the paper is as follows. In Section 2 we recall some necessary background on module categories over fusion categories, exact sequences of fusion categories, and the class of group-theoretical fusion categories. In Section 3 we introduce and study