Journal of Algebra 481 (2017) 1–11

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Locally finite groups and their subgroups with small centralizers ☆

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ARTICLE INFO

Article history:
Received 25 June 2016
Available online 4 March 2017
Communicated by E.I. Khukhro

MSC:
20F50
20E36

Keywords:
Locally finite group
Centralizer

ABSTRACT

Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$. We show that G is almost locally soluble (Theorem 1.1). The key step in the proof is the following characterization of $PSL_p(k)$: An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(A)$ involves no infinite simple groups for any $a \in A^{\#}$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.

☆ This work was carried out during the third author's visits to Mimar Sinan Fine Arts University and Middle East Technical University with financial support from MSGSÜ and TÜBİTAK. He thanks MSGSÜ and TÜBİTAK for support and the visited universities for hospitality.

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http://dx.doi.org/10.1016/j.jalgebra.2017.02.018
0021-8693/© 2017 Published by Elsevier Inc.

1. Introduction

A group is locally finite if every finite subset of the group generates a finite subgroup. In the theory of locally finite groups centralizers play an important role. In particular the following family of problems has attracted great deal of attention in the past. Let G be a locally finite group containing a finite subgroup A such that $C_G(A)$ is small in some sense. What can be said about the structure of G? In some situations quite significant information about G can be deduced. For example if $|A| = 2$ and $C_G(A)$ is finite, then G has a nilpotent subgroup of class at most two with finite index bounded by a function of $|C_G(A)|$ [19]. If G contains an element of prime order p whose centralizer is finite of order m, then G contains a nilpotent subgroup of finite (m, p)-bounded index and p-bounded nilpotency class. This result for locally nilpotent periodic groups is due to Khukhro [25] while the reduction to the nilpotent case was obtained combining a result of Hartley and Meixner [20] with that of Fong [8]. The latter uses the classification of finite simple groups. Another important result in this direction is Hartley's theorem that if G has an element of order n with finite centralizer of order m, then G contains a locally soluble subgroup with finite (m, n)-bounded index [13]. The interested reader should consult two excellent survey articles due to Hartley [14,16] and the paper [3] due to Belyaev and Hartley for the comprehensive description of the developments in this area in the twentieth century.

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasicyclic p-groups, or Prüfer p-groups). By a deep result obtained independently by Shunkov [32] and Kegel and Wehrfritz [23] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. In the literature there are many results on Chernikov centralizers in locally finite groups. By and large, they resemble the corresponding results on finite centralizers. In particular, Hartley proved in [15] that if a locally finite group contains an element of prime-power order with Chernikov centralizer, then it is almost locally soluble. A group is said to almost have certain property if it contains a subgroup of finite index with that property.

Infinite locally finite groups containing a non-cyclic subgroup with finite centralizer can be simple. One example is provided by the group $PSL(2, k)$, where k is an infinite locally finite field of odd characteristic. This group contains a non-cyclic subgroup of order four with finite centralizer. In [30] the third author proved that if a locally finite group G contains a non-cyclic subgroup A of order p^2 for a prime p such that $C_G(A)$ is finite and $C_G(a)$ has finite exponent for all $a \in A^\#$, then G is almost locally soluble and has finite exponent. Here the symbol $A^\#$ stands for the set of the nontrivial elements of A.

If G and T are groups, we say that G involves T if there are subgroups $K \leq H \leq G$, with K normal in H, such that $H/K \cong T$. The main purpose of the present article is to prove the following theorem.