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The strong global dimension of piecewise hereditary algebras [☆]

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A B S T R A C T

Let A be a finite-dimensional piecewise hereditary algebra over an algebraically closed field. This text investigates the strong global dimension of A. This invariant is characterised in terms of the lengths of sequences of tilting mutations relating A to a hereditary abelian category, in terms of the generating hereditary abelian subcategories of the derived category of A, and in terms of the Auslander–Reiten structure of that derived category.

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Auslander–Reiten theory
Tilting theory

0. Introduction

Let A be a finite-dimensional algebra over an algebraically closed field $\Bbbk$. Its category of finitely generated (left) modules is denoted by $\operatorname{mod} A$. Then, A is called *piecewise hereditary* if the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ is equivalent as a triangulated category to $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H}$ is a hereditary abelian ($\Bbbk$-linear) category with split idempotents, finite-dimensional Hom-spaces, and with tilting objects. In the particular case where $A \simeq \operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ for some tilting object $T \in \mathcal{H}$, the algebra A is called *quasi-tilted*. It is called *tilted* when, in addition, $\mathcal{H} \simeq \operatorname{mod} H$ for some finite-dimensional hereditary algebra H. In [13], Happel proved that a hereditary abelian category as above is equivalent to the category of finitely generated modules over a hereditary algebra or to the category of coherent sheaves over a weighted projective line [10].

In the representation theory of finite-dimensional algebras, piecewise hereditary algebras play a particular and important role. On the one hand, this is due to information that is already known on the representation theory of hereditary algebras, of tilted algebras (see [16]) or of quasi-tilted algebras of canonical type (see [24]), and also to Happel's description of the bounded derived category of hereditary abelian categories (see below). On the other hand, these algebras are used in many parts of representation theory. For instance, in order to develop the representation theory of other classes of algebras such as the selfinjective algebras (see [28]) or the cluster tilted algebras (see [4]), in order to investigate singularity theory (see [20]), or in order to categorify cluster algebras (see [8]).

The homological characterisation of quasi-tilted algebras [14] and the Liu–Skowroński criterion for tilted algebras (see [5]) suggest that the quasi-tilted algebras are the closest piecewise hereditary algebras to hereditary ones, and it is the main objective of this text to give theoretical and numerical criteria to determine how far a piecewise hereditary algebra is from being hereditary.

Recall the description of $\mathcal{D}^b(\mathcal{H})$ made by Happel in [11]: Any object is the direct sum of (finitely many) stalk complexes $X[i]$ ($X \in \mathcal{H}$ and $i \in \mathbb{Z}$); And, for every $i, j \in \mathbb{Z}$ and $X, Y \in \mathcal{H}$, the morphism space $\operatorname{Hom}(X[i], Y[j])$ is naturally isomorphic to $\operatorname{Hom}_{\mathcal{H}}(X, Y)$ if $i = 0$, to $\operatorname{Ext}^1_{\mathcal{H}}(X, Y)$ if $i = 1$, and is equal to zero otherwise. Hence, when $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{D}^b(\mathcal{H})$, then there exists a tilting object $T \in \mathcal{D}^b(\mathcal{H})$ (that is, an object such that $\operatorname{Hom}(T, T[i]) = 0$ for $i \in \mathbb{Z}\backslash\{0\}$, and such that $\mathcal{D}^b(\mathcal{H})$ is the smallest full triangulated subcategory of $\mathcal{D}^b(\mathcal{H})$ containing T and stable under taking direct summands) such that $A \simeq \operatorname{End}(T)^{\mathrm{op}}$ as $\Bbbk$-algebras. In particular, there exists a minimal $\ell \in \mathbb{N}$ and there exists $s \in \mathbb{Z}$ such that T lies in the additive closure $\bigvee_{i=0}^{\ell} \mathcal{H}[s+i]$ of the union $\bigcup_{i=0}^{\ell} \mathcal{H}[s+i]$. When $\ell = 0$, then A is quasi-tilted. And one may expect that the larger ℓ, the further A is from being quasi-tilted. Note however that there exist examples where $\ell = 1$ and A is hereditary (see [12]).