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Trace formula for crystals with group actions

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We prove a cohomological trace formula of crystals equipped with group actions.

1. Introduction and statement of the main results

In this paper, let k be a finite field of q-elements and let A be a k-algebra. All tensors are over k and all schemes are assumed to be noetherian and separated over k, unless otherwise specified. For any scheme X, let $\sigma_X : X \to X$ be the morphism induced by the q-th power map on $\mathcal{O}_X$. Let G be a finite group.

Definition 1.1. (1) A coherent τ-sheaf over A on X is a pair $(\mathcal{F}, \tau_{\mathcal{F}})$ consisting of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_X \otimes A$-linear homomorphism

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$$\tau_{\mathcal{F}} : \bigoplus_{n\geq 1}(\sigma_X^n)^*\mathcal{F}\otimes A \to \mathcal{F}\otimes A$$

such that the induced homomorphism $(\sigma_X^n)^*\mathcal{F}\otimes A \to \mathcal{F}\otimes A$ is a zero map for $n \gg 0$.

(2) A homomorphism $(\mathcal{F}, \tau_{\mathcal{F}}) \to (\mathcal{G}, \tau_{\mathcal{G}})$ is an $\mathcal{O}_X$-linear map $\phi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{n\geq 1}(\sigma_X^n)^*\mathcal{F}\otimes A & \xrightarrow{\tau_{\mathcal{F}}} & \mathcal{F}\otimes A \\ \downarrow{\scriptstyle \oplus(\sigma_X^n)^*\phi\otimes \mathrm{id}_A} & & \downarrow{\scriptstyle \phi\otimes\mathrm{id}_A} \\ \bigoplus_{n\geq 1}(\sigma_X^n)^*\mathcal{G}\otimes A & \xrightarrow{\tau_{\mathcal{G}}} & \mathcal{G}\otimes A. \end{array}$$

(3) A coherent τ-sheaf $(\mathcal{F}, \tau_{\mathcal{F}})$ is called nilpotent if there exists a decreasing filtration $\{\mathcal{F}_m\}_{m\in\mathbb{N}}$ of $\mathcal{F}$ by coherent submodules such that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_m = 0$ for $m \gg 0$ and

$$\tau_{\mathcal{F}}\Big(\bigoplus_{n\geq 1}(\sigma_X^n)^*\mathcal{F}_m\otimes A\Big) \subset \mathcal{F}_{m+1}\otimes A \text{ for any } m.$$

Definition 1.2. (1) Let $\mathrm{Coh}_\tau(X, A)$ be the category of coherent τ-sheaves over A on X and let $\mathrm{NilCoh}_\tau(X, A)$ be its full subcategory consisting of nilpotent objects.

(2) Let $\mathrm{Coh}_\tau^G(X, A)$ (resp. $\mathrm{NilCoh}_\tau^G(X, A)$) be the category of coherent τ-sheaves (resp. nilpotent τ-sheaves) over A on X with G-actions.

Definition 1.3. For any abelian category $\mathcal{A}$, let $K_0(\mathcal{A})$ be the Grothendieck group of $\mathcal{A}$ generated by the isomorphic classes $[M]$ of objects in $\mathcal{A}$, modulo relations $[M_2] = [M_1] + [M_3]$ for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathcal{A}$.

Definition 1.4. The category $\mathrm{Coh}_\tau^G(X, A)$ is abelian and its full subcategory $\mathrm{NilCoh}_\tau^G(X, A)$ is stable under sub-quotients, extensions and isomorphisms. Then we can define the quotient category

$$\mathrm{Crys}^G(X, A) := \frac{\mathrm{Coh}_\tau^G(X, A)}{\mathrm{NilCoh}_\tau^G(X, A)}.$$

Objects in $\mathrm{Crys}^G(X, A)$ are called crystals.

A morphism α in $\mathrm{Coh}_\tau^G(X, A)$ is called a nil-isomorphism if $\ker(\alpha)$ and $\mathrm{coker}(\alpha)$ are nilpotent.