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Quasi-shuffle products revisited

Michael E. Hoffman [a,*], Kentaro Ihara [b]

[a] U.S. Naval Academy, Annapolis, MD 21402 USA
[b] Kindai University, 3-4-1 Kowakae, Higashi-Osaka 577-8502, Japan

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ABSTRACT

Quasi-shuffle products, introduced by the first author, have been useful in studying multiple zeta values and some of their analogues and generalizations. The second author, together with Kajikawa, Ohno, and Okuda, significantly extended the definition of quasi-shuffle algebras so it could be applied to multiple q-zeta values. This article extends some of the algebraic machinery of the first author's original paper to the more general definition, and demonstrates how various algebraic formulas in the quasi-shuffle algebra can be obtained in a transparent way. Some applications to multiple zeta values, interpolated multiple zeta values, multiple q-zeta values, and multiple polylogarithms are given.

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1. Introduction

This article revisits the construction of quasi-shuffle products in [7]. In [12] the construction of [7] was put in a more general setting that had two chief advantages: (i) it simultaneously applied to multiple zeta and multiple zeta-star values and their extensions; and (ii) it could be applied to the q-series version of multiple zeta values studied in [4]. Here we show that some of the algebraic machinery developed in [7], particularly

* Corresponding author.
E-mail addresses: meh@usna.edu (M.E. Hoffman), k-ihara@math-kindai.ac.jp (K. Ihara).

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the coalgebra structure and linear functions induced by formal power series (not considered in [12]), can be carried over to the more general setting and used to make the calculations in the quasi-shuffle algebra, including many of [12], more transparent. We also describe some applications of quasi-shuffle algebras not considered in [12], including applications to the interpolated multiple zeta values introduced in [16].

The original quasi-shuffle product was inspired by the multiplication of multiple zeta values, i.e.,

$$\sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}, \tag{1}$$

with $i_1 > 1$ to insure convergence. One can associate to the series (1) the monomial $z_{i_1}\cdots z_{i_k}$ in the noncommuting variables $z_1, z_2, \ldots$; then we write the value (1) as $\zeta(z_{i_1}\cdots z_{i_k})$. For any monomials $w = z_i w'$ and $v = z_j v'$, define the product $w * v$ recursively by

$$w * v = z_i(w' * v) + z_j(w * v') + z_{i+j}(w' * v'). \tag{2}$$

Then $\zeta(w)\zeta(v) = \zeta(w*v)$, where we think of ζ as a linear function on monomials. As we shall see in the next section, the recursive rule (2) is a quasi-shuffle product on monomials in the z_i derived from the product $\diamond$ on the vector space of z_i's given by $z_i \diamond z_j = z_{i+j}$.

In [4] the multiple q-zeta values were defined as

$$\sum_{n_1>\cdots>n_k\geq 1} \frac{q^{(i_1-1)n_1}\cdots q^{(i_k-1)n_k}}{[n_1]_q^{i_1}\cdots [n_k]_q^{i_k}}, \tag{3}$$

where $[n]_q = 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$. If we denote (3) by $\zeta_q(z_{i_1}\cdots z_{i_k})$, then to have $\zeta_q(w)\zeta_q(v) = \zeta_q(w*v)$ the recursion (2) must be significantly modified: in place of $z_i \diamond z_j = z_{i+j}$ we must have

$$z_i \diamond z_j = z_{i+j} + (1-q)z_{i+j-1}.$$

This means that to have a theory of quasi-shuffle algebras that applies to multiple q-zeta values, two restrictions in the original construction of [7] must be removed: that the product $a \diamond b$ of two letters be a letter, and that the operation $\diamond$ preserve a grading. This was done in [12]. The same paper also addressed the relation between multiple zeta values (1) and the multiple zeta-star values

$$\zeta^\star(z_{i_1}\cdots z_{i_k}) = \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}. \tag{4}$$

This relation can be expressed in terms of a linear isomorphism (here denoted Σ) from the vector space of monomials in the z_i's to itself. The function Σ acts on monomials as, e.g.,

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