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Examples of non-commutative crepant resolutions of Cohen Macaulay normal domains

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ABSTRACT

Let A be a Cohen–Macaulay normal domain. A non-commutative crepant resolution (NCCR) of A is an A-algebra Γ of the form $\Gamma = \operatorname{End}_A(M)$, where M is a reflexive A-module, Γ is maximal Cohen–Macaulay as an A-module and $\operatorname{gldim}(\Gamma)_P = \dim A_P$ for all primes P of A. We give bountiful examples of equi-characteristic Cohen–Macaulay normal local domains and mixed characteristic Cohen–Macaulay normal local domains having NCCR. We also give plentiful examples of affine Cohen–Macaulay normal domains having NCCR.

1. Introduction

Let A be a Cohen–Macaulay normal domain. Van den Bergh [14] defined *a non-commutative crepant resolution* of A (henceforth NCCR) to be an A-algebra Γ of the form $\Gamma = \operatorname{End}_A(M)$, where M is a reflexive A-module, Γ is maximal Cohen–Macaulay as an A-module and $\operatorname{gldim}(\Gamma)_P = \dim A_P$ for all primes P of A. We should remark that

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Van den Bergh only defined this for Gorenstein normal domains as this has applications in algebraic geometry. However there are many algebraic reasons for consider this generalization, see [4]. For a nice survey on this topic see [10]. In general, it is subtle to construct NCCR's. In this paper we give bountiful examples of Cohen–Macaulay normal domains having a NCCR.

1.1. *Mixed characteristic case:* We now outline in brief our construction. Recall $f \in \mathbb{Z}[X_1, \dots, X_n]$ has content 1 if 1 belongs to the ideal generated by the coefficients of f. We say f is $\mathbb{Q}$-smooth if $\mathbb{Q}[X_1, \dots, X_n]/(f)$ is a regular ring. For a prime p we say f is smooth mod-p if $\mathbb{Z}_p[X_1, \dots, X_n]/(f)$ is a regular ring. It is well-known that if f is $\mathbb{Q}$-smooth then is smooth mod-p for infinitely many primes p. Our result is:

Theorem 1.2. *Let $(A, \mathfrak{m})$ be an excellent normal Cohen–Macaulay local domain of mixed characteristic with perfect residue field $k = A/\mathfrak{m}$ of characteristic $p > 0$. Assume A has a NCCR and that $\dim A \geq 2$. Also assume that A has a canonical module. Let $f \in \mathbb{Z}[X_1, \dots, X_n]$ be of content 1. Also assume that f is $\mathbb{Q}$-smooth and is smooth mod-p. Set $T = A[X_1, \dots, X_n]/(f)$ and let $\mathfrak{n}$ be a maximal ideal of T containing $\mathfrak{m}T$. Set $A(f) = T_{\mathfrak{n}}$. Then*

- (i) *$A(f)$ is flat over A with regular fiber. In particular if A is Gorenstein then so is $A(f)$.*
- (ii) *$A(f)$ is an excellent normal Cohen–Macaulay local domain of mixed characteristic with perfect residue field.*
- (iii) *$A(f)$ has a NCCR.*

Furthermore if $\Gamma = \operatorname{Hom}_A(M, M)$ is a NCCR of A then $\Lambda = \Gamma \otimes_A A(f)$ is a NCCR of $A(f)$.

1.3. Two dimensional rings of finite representation type have a NCCR (see [9, Theorem-6]). For examples of two dimensional mixed characteristic rings of finite representation type see [12]. Using the above recipe we can construct plentiful examples of Cohen–Macaulay local domain of mixed characteristic having NCCR's. If k is algebraically closed then it can be easily shown that if $A(f) \cong A(g)$ as A-algebra's then the hypersurfaces defined by f and g in $\mathbb{A}^n(k)$ are birational.

1.4. *Equi-characteristic case (local):* Let $(A, \mathfrak{m})$ be an excellent equi-characteristic Cohen–Macaulay local domain with perfect residue field k. Assume A contains k, $\dim A \geq 2$ and that it has a canonical module. Let $f \in k[X_1, \dots, X_n]$ be smooth, i.e., $k[X_1, \dots, X_n]/(f)$ is a regular ring. We show

Theorem 1.5. *(with hypotheses as in 1.4) Assume A has a NCCR. Set $T = A[X_1, \dots, X_n]/(f)$. Let $\mathfrak{n}$ be a maximal ideal of T containing $\mathfrak{m}T$. Set $A(f) = T_{\mathfrak{n}}$. Then*