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Lifting of elements of Weyl groups

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ABSTRACT

Suppose G is a reductive algebraic group, T is a Cartan subgroup of G, $N = \mathrm{Norm}(T)$, and $W = N/T$ is the Weyl group. If $w \in W$ has order d, it is natural to ask about the orders lifts of w to N. It is straightforward to see that the minimal order of a lift of w has order d or $2d$, but it can be a subtle question which holds. We first consider the question of when W itself lifts to a subgroup of N (in which case every element of W lifts to an element of N of the same order). We then consider two natural classes of elements: regular and elliptic. In the latter case all lifts of w are conjugate, and therefore have the same order. We also consider the twisted case.

1. Introduction

Let G be a connected reductive group over an algebraically closed field F. Choose a Cartan subgroup $T \subset G$, let $N = \mathrm{Norm}_G(T)$ be its normalizer, and let $W = N/T$ be the Weyl group. We have the exact sequence

$$1 \to T \to N \xrightarrow{p} W \to 1. \tag{1.1}$$

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It is natural to ask what can be said about the orders of lifts of an element $w \in W$ to N. What is the smallest possible order of a lift of w? In particular, can w be lifted to an element of N of the same order?

Write $o(*)$ for the order of an element of a group, and let $N_w = p^{-1}(w) \subset N$. Define

$$\tilde{o}(w, G) = \min_{g \in N_w} o(g). \tag{1.2a}$$

The most important case is for the adjoint group G_{ad}, so define

$$\tilde{o}_{\mathrm{ad}}(w) = \tilde{o}(w, G_{\mathrm{ad}}). \tag{1.2b}$$

It is clear that $\tilde{o}(w, G)$ only depends on the conjugacy class $\mathcal{C}$ of w, so write $\tilde{o}(\mathcal{C}, G)$ and $\tilde{o}_{\mathrm{ad}}(\mathcal{C})$ accordingly.

An essential role is played by the Tits group. This is a group which fits in an exact sequence $1 \to T_2 \to \mathcal{T} \to W \to 1$ where T_2 is a certain subgroup of the elements of T of order (1 or) 2. This implies $\tilde{o}(w, G) = o(w)$ or $2o(w)$, but it can be difficult to determine which case holds.

We also consider the twisted situation. Let δ be an automorphism of G of finite order which preserves a pinning, and set ${}^{\delta}G = G \rtimes \langle \delta \rangle$. Let ${}^{\delta}N = \mathrm{Norm}_{{}^{\delta}G}(T)$ and ${}^{\delta}W = {}^{\delta}N/T$. Then conjugacy in $W\delta$ is the same as δ-twisted conjugacy in W, and we can ask about the order of lifts of elements of $W\delta$ to ${}^{\delta}N$. See Section 2 for details.

We say W lifts to G if the exact sequence (1.1) splits, in which case $\tilde{o}(w, G) = o(w)$ for all w. If this is not the case, it may not be practical to give a formula for $\tilde{o}(w, G)$ for all conjugacy classes. Rather, this can be done for several natural families. We say $w \in W\delta$ is elliptic if it has no nontrivial fixed vectors in the reflection representation. It is well known that this implies all lifts of w are T-conjugate. (If the coset gT maps to w, then for $t \in T$, $t(gT)t^{-1} = tw(t^{-1})(gT)$; if w is elliptic the map $t \to tw(t^{-1})$ has finite kernel, hence is surjective.) Therefore all lifts of an elliptic element have the same order $\tilde{o}(w, G)$. An element $w \in W\delta$ is said to be regular if it has a regular eigenvector (see Section 7 and [15]).

Let $\rho^{\vee}$ be one-half the sum of the positive coroots in any positive system. We refer to the element $z_G = (2\rho^{\vee})(-1)$ as the *principal involution* in G. It is contained in the center $Z(G)$, is independent of the choice of positive system, and is fixed by every automorphism of G.

Here is a result concerning when W lifts, so $\tilde{o}(w, G) = o(w)$ for all w.

Theorem A. *If the characteristic of F is 2, then the Tits group $\mathcal{T}$ is isomorphic to the Weyl group, so the exact sequence (1.1) splits.*