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# The monoidal structure on strict polynomial functors

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A R T I C L E I N F O

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A B S T R A C T

The category of strict polynomial functors inherits an internal tensor product from the category of divided powers. To investigate this monoidal structure, we consider the category of representations of the symmetric group $\mathfrak{S}_d$ which admits a tensor product coming from its Hopf algebra structure. It is classical that there exists a functor $\mathcal{F}$ from the category of strict polynomial functors to the category of representations of the symmetric group. Our main result is that this functor $\mathcal{F}$ is monoidal. In addition we study the relations under $\mathcal{F}$ between projective strict polynomial functors and permutation modules and the link to symmetric functions.



## 1. Introduction

Strict polynomial functors were first defined by Friedlander and Suslin in [1], using polynomial maps of finite dimensional vector spaces over a field $k$. They showed that the category of strict polynomial functors of a fixed degree $d$ is equivalent to the category of modules over the Schur algebra $S_k(n, d)$ whenever $n \geq d$. These algebras, named after

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Issai Schur and originally used to describe the polynomial representations of the general linear group, have been extensively investigated.

Another description of strict polynomial functors, namely defining them as $k$-linear representations of the category of divided powers, provides an important structure on the category of strict polynomial functors and hence on modules over the Schur algebra: Via Day convolution, the category of strict polynomial functors inherits a closed symmetric monoidal structure from the category of divided powers. This particular tensor product can be implicitly found in works by Chałupnik [2] and Touzé [3]; an explicit definition is given by Krause in [4].

In order to describe this tensor product more explicitly we make use of the monoidal structure on the category of representations of the symmetric group. In fact, there exists a functor $\mathcal{F}$, sometimes called the Schur functor, from the category of strict polynomial functors to the category of representations of the symmetric group which allows us to compare these monoidal structures. The main result of this paper is that the monoidal structure is preserved under the functor $\mathcal{F}$. This yields explicit formulae for the tensor product of certain polynomial functors.

We work over an arbitrary commutative ring $k$. We start by recalling some basic definitions concerning strict polynomial functors and the description of the internal tensor product given in [4]. In the second section we focus our attention on representations of the symmetric group $\mathfrak{S}_d$. In particular, we consider the $k\mathfrak{S}_d$-module structure on the $d$-th tensor power of a free $k$-module $E$ and its decomposition into permutation modules. We calculate the tensor product of permutation modules and its decomposition.

In the fourth section we show that the functor $\mathcal{F}$ maps certain important projective objects in the category of strict polynomial functors to the permutation modules. For this an essential ingredient is the parametrization of morphisms of these objects given by Totaro [5]. Finally we prove that $\mathcal{F}$ is a monoidal functor, see Theorem 4.4.

In the last section we assume that $k$ is a field of characteristic 0. In this case, the functor $\mathcal{F}$ induces an equivalence between strict polynomial functors and representations of the symmetric group. Moreover we explain and use the connection to symmetric functions.

## 2. Strict polynomial functors

In the following we briefly recall the definition of strict polynomial functors and of the internal tensor product as described in [4]. Let $k$ be a commutative ring and denote by $\mathsf{P}_k$ the category of finitely generated projective $k$-modules. For $d \in \mathbb{N}$ and $V \in \mathsf{P}_k$ denote by $\Gamma^d V$ the $\mathfrak{S}_d$-invariant part of $V^{\otimes d}$ where the (right) action of $\mathfrak{S}_d$ is given by permuting the factors. Sending a module $V$ to $\Gamma^d V$ yields a functor $\Gamma^d : \mathsf{P}_k \to \mathsf{P}_k$.

We define the *category of degree $d$ divided powers* $\Gamma^d \mathsf{P}_k$ which has the same objects as $\mathsf{P}_k$ and where the morphisms between two objects $V$ and $W$ are given by

$$\operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(V, W) := \Gamma^d \operatorname{Hom}(V, W) = (\operatorname{Hom}(V, W)^{\otimes d})^{\mathfrak{S}_d}.$$