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Words and characters in finite p-groups ☆

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Given a group word w in k variables, a finite group G and $g \in G$, we consider the number $N_{w,G}(g)$ of k-tuples $(g_1, \dots, g_k)$ of elements of G such that $w(g_1, \dots, g_k) = g$. In this work we study the functions $N_{w,G}$ for the class of nilpotent groups of nilpotency class 2. We show that, for the groups in this class, $N_{w,G}(1) \geq |G|^{k-1}$, an inequality that can be improved to $N_{w,G}(1) \geq |G|^k/|G_w|$ (G_w is the set of values taken by w on G) if G has odd order. This last result is explained by the fact that the functions $N_{w,G}$ are characters of G in this case. For groups of even order, all that can be said is that $N_{w,G}$ is a generalized character, something that is false in general for groups of nilpotency class greater than 2. We characterize group theoretically when $N_{x^n,G}$ is a character if G is a 2-group of nilpotency class 2. Finally we also address the (much harder) problem of studying if $N_{w,G}(g) \geq |G|^{k-1}$ for $g \in G_w$, proving that this is the case for the free p-groups of nilpotency class 2 and exponent p.

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1. Introduction

Given a group word w in k variables $x_1, \ldots, x_k$, that is an element in the free group F_k on $x_1, \ldots, x_k$, we can define for any k elements $g_1, \ldots, g_k$ in a group G the element $w(g_1, \ldots, g_k) \in G$ by applying to w the group homomorphism from F_k to G sending x_i to g_i. For G a finite group and $g \in G$ we consider the number

$$N_{w,G}(g) = |\{(g_1, \ldots, g_k) \in G^{(k)} \mid w(g_1, \ldots, g_k) = g\}|. \tag{1}$$

($G^{(k)}$ denotes the k-fold cartesian product of G with itself.) So $N_{w,G}(g)$ can be thought of as the number of solutions of the equation $w = g$. The set of word values of w on G, i.e., the set of elements $g \in G$ such that the equation $w = g$ has a solution in $G^{(k)}$, is denoted by G_w.

There is an extensive literature on the functions $N_{w,G}$, sometimes expressed in terms of the probabilistic distribution $P_{w,G} = N_{w,G}/|G|^k$. For example Nikolov and Segal gave in [10] a characterization of the finite nilpotent (and also solvable) groups based on these probabilities: G is nilpotent if and only if $\inf_{w,g} P_{w,G}(g) > p^{-|G|}$, where w and g range over all words and G_w, respectively, and p is the largest prime dividing $|G|$. One of the implications is easy: if G is not nilpotent the infimum is in fact zero. Indeed, we can consider the k-th left-normed lower central word $\gamma_k = [[[x_1, x_2], x_3], ..., x_k]$. Since G is not nilpotent, there exists some non-trivial element $g \in G_{\gamma_k}$ (for any k). Since $\gamma_k(g_1, \ldots, g_k) = 1$ if some $g_i = 1$, we have that $P_{\gamma_k,G}(g) \leq (|G| - 1)^k/|G|^k$, which can be made arbitrarily small. On the other hand the estimation $P_{w,G}(g) > p^{-|G|}$ for $g \in G_w$ seems to be far from sharp and Amit in an unpublished work has conjectured that if G is nilpotent, $P_{w,G}(1) \geq 1/|G|$.

We prefer to give our results in terms of the functions $N_{w,G}$. In this paper we focus our attention on finite nilpotent groups of nilpotency class 2, which we take to be p-groups right from the outset, so all the results quoted here are referred to this family of groups. In Section 2 we consider a natural equivalence relation for words that enable us to assume that they have a special simple form. Then it is not difficult to prove Amit's conjecture $N_{w,G}(1) \geq |G|^{k-1}$ for $w \in F_k$. This result has been proved independently by Levy in [7] using a similar procedure, although our approach to the concept of word equivalence is different. In the next two sections we show that the functions $N_{w,G}$ are generalized characters, a result that is false for nilpotent groups of nilpotency class greater than 2, and even more, if G has odd order, they are genuine characters. In particular we obtain an improvement of Amit's conjectured bound, namely, $N_{w,G}(1) \geq |G|^k/|G_w|$. For 2-groups, there are easy examples where $N_{x^2,G}$ fails to be a character and we actually characterize group-theoretically when this happens for the power words $w = x^n$ (always within the class of nilpotent 2-groups of nilpotency class 2). In the last section we consider briefly the conjecture $N_{w,G}(g) \geq |G|^{k-1}$ for $w \in F_k$ and $g \in G_w$. This problem is much harder than the case $g = 1$ and only some partial results have been obtained, for instance confirming the conjecture if G is a free nilpotent p-group of nilpotency class 2 and exponent p.