Journal of Algebra 485 (2017) 332–352

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# The third order modular linear differential equations

Masanobu Kaneko [a,*,1], Kiyokazu Nagatomo [b,2], Yuichi Sakai [c]

[a] *Faculty of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan*
[b] *Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan*
[c] *Yokomizo 3012-2, Oki-machi, Mizuma-gun, Fukuoka 830-0405, Japan*

A R T I C L E   I N F O

*Article history:*
Received 13 March 2015
Available online 17 May 2017
Communicated by Nicolás Andruskiewitsch

*MSC:*
primary 11F11, 81T40
secondary 17B69

*Keywords:*
Vertex operator algebra
Modular invariance
Modular linear differential equation
$n$-dimensional conformal field theory

A B S T R A C T

We propose a third order generalization of the Kaneko–Zagier modular differential equation, which has two parameters. We describe modular and quasimodular solutions of integral weight in the case where one of the exponents at infinity is a multiple root of the indicial equation. We also classify solutions of "character type", which are the ones that are expected to relate to characters of simple modules of vertex operator algebras and one-point functions of two-dimensional conformal field theories. Several connections to generalized hypergeometric series are also discussed.



* Corresponding author.
*E-mail addresses:* mkaneko@math.kyushu-u.ac.jp (M. Kaneko), nagatomo@math.sci.osaka-u.ac.jp (K. Nagatomo), dynamixaxs@gmail.com (Y. Sakai).

[1] This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 23340010.
[2] This work was supported by Japan Society for the Promotion of Science, Grant-in-Aid for Challenging Exploratory Research 26610007. The second author was also partially supported by Max-Planck Institute für Mathematik.

http://dx.doi.org/10.1016/j.jalgebra.2017.05.007
0021-8693/

## 1. Introduction

This paper studies a third order generalization of the Kaneko–Zagier equation (K–Z equation for short), which is called the *third order K–Z equation* here. The K–Z equation first appeared in [11] in connection with supersingular $j$-invariants of elliptic curves, and subsequently, various modular and quasimodular solutions of the K–Z equation were found and studied in [5–8], etc. One of the characteristic properties of the K–Z equation is the invariance of the space of solutions under the standard slash action of the modular group $\Gamma_1 = SL_2(\mathbb{Z})$, and indeed, our generalization has the same property.

Under mild conditions on the coefficient functions, we determine in §2 the form of what we call the third order *modular linear differential equation* as

$$f''' - \frac{k+2}{4}E_2(\tau)f'' + \left\{\frac{(k+1)(k+2)}{4}E_2'(\tau) + \alpha E_4(\tau)\right\} f'(\tau) \\ - \left\{\frac{k(k+1)(k+2)}{24}E_2''(\tau) + \frac{k\alpha}{4}E_4'(\tau) - \beta E_6(\tau)\right\} f(\tau) = 0, \quad (1)$$

where $\tau$ is a variable in the complex upper half-plane $\mathbb{H}$, and $'$ is the Euler operator of $q\,(= e^{2\pi\sqrt{-1}\tau})$ (see Theorem 1 in §2). The $E_k(\tau)$ is the *normalized* Eisenstein series of weight $k$ given by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where $B_k$ is the $k$th Bernoulli number and $\sigma_m(n)$ is the sum of $m$th powers of positive divisors of $n$. The parameter $k$ is expected to stand for the weight of $f$, and $\alpha, \beta$ are complex parameters.

As shown in [8], the K–Z equation is closely related to two-dimensional conformal field theory (2DCFT for short). The papers [1] and [2,3], which may be viewed as companion papers of the present one, study affine 2DCFT with at most 20 simple modules or 5 independent pseudo-characters and the minimal models with at most four simple modules, respectively. The (formal) characters of such 2DCFT were expected to satisfy one of the third order K–Z equations.

One of the main results in this paper (given in §3) is an almost complete description of modular and quasimodular solutions in the case where the indicial equation of (1) with $\beta = 0$ at $q = 0$ has a multiple root and $k$ is an integer (Theorem 2 in §3.1, Theorem 4 in §3.2). The other is the determination of solutions of *character type* (for the definition see the beginning of §4), which is characterized by integrality and positivity of Fourier coefficients of an associated weight 0 function (Proposition 6 and Theorem 7 in §4). As motivated by [10], we discuss in §5 a relation between the third order K–Z equation and hypergeometric series (Theorem 8 in §5).