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Hilbert series of symmetric ideals in infinite polynomial rings via formal languages [☆]

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A B S T R A C T

Let R be the polynomial ring $K[x_{i,j}]$ where $1 \leq i \leq r$ and $j \in \mathbf{N}$, and let I be an ideal of R stable under the natural action of the infinite symmetric group S_∞. Nagel–Römer recently defined a Hilbert series $H_I(s,t)$ of I and proved that it is rational. We give a much shorter proof of this theorem using tools from the theory of formal languages and a simple algorithm that computes the series.

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1. Introduction

1.1. Statement of results

Let R be the polynomial ring over the field K in variables $x_{i,j}$, where $i \in \{1, \dots, r\}$ and $j \in \mathbf{N} = \{0, 1, 2, \dots\}$. The infinite symmetric group S_∞ acts on R (by fixing the first index and moving the second), and a fundamental result, proved originally by Cohen [2] but subsequently rediscovered [1,4], is that R is S_∞-noetherian: that is, any S_∞-ideal in R is generated by the S_∞-orbits of finitely many elements. Given this, one can begin to study finer properties of ideals. In this paper, we investigate their Hilbert series. Our technique gives a new, more natural, and far less technical handle on the Hilbert series than the methods of the original proof of rationality of the series.

Let $I \subset R$ be a homogeneous S_∞-ideal. For $n \geq 0$, let $R_n \subset R$ be the subalgebra generated by the variables $x_{i,j}$ with $1 \leq i \leq r$ and $j \leq n$, and put $I_n = I \cap R_n$. Then I_n is a finitely generated graded R_n-module, and so its Hilbert series $H_{I_n}(t)$ is a well-defined rational function. We define the Hilbert series of I by

$$H_I(s,t) = \sum_{n \geq 0} H_{I_n}(t) s^n.$$

This series was introduced by Nagel–Römer [7], who proved the following theorem:

Theorem 1.1. *The series $H_I(s,t)$ is a rational function of s and t.*

The purpose of this paper is to give a new proof of this theorem. Our proof is shorter and (in our opinion) conceptually clearer than the one given in [7].

Remark 1.2. In fact, [7] work with what we would call $H_{R/I}(s,t)$, but it is a trivial matter to pass between this and our $H_I(s,t)$, since the sum of these two series is $H_R(s,t)$. Note that R_n is a polynomial ring in $r(n+1)$ variables, so $H_{R_n}(t) = (1-t)^{-r(n+1)}$, and thus

$$H_R(s,t) = \frac{1}{(1-t)^r - s} \quad \square$$