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Exceptional representations of Weyl groups

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ABSTRACT

We consider various consequences of the existence of exceptional representations of an irreducible Weyl group.

1.1. Let W be a finite, irreducible Coxeter group and let S be the set of simple reflections of W; let $l : W \to \mathbf{N}$ be the length function. Let $\mathrm{Irr}W$ be a set of representatives for the isomorphism classes of irreducible representations of W over $\mathbf{C}$, the complex numbers. Let $\mathcal{A} = \mathbf{C}[v, v^{-1}]$, $\mathcal{A}' = \mathbf{C}[v^2, v^{-2}]$ (v an indeterminate). We have $\mathcal{A}' \subset \mathcal{A} \subset K \supset K' \supset \mathcal{A}'$ where $K = \mathbf{C}(v)$, $K' = \mathbf{C}(v^2)$. Let H be the Hecke algebra over $\mathcal{A}$ associated to W; thus H has generators T_s ($s \in S$) and generators $(T_s+1)(T_s-v^2) = 0$ for $s \in S$, $T_sT_{s'}T_s \cdots = T_{s'}T_sT_{s'} \dots$ for $s \neq s'$ in S (both products have m factors where m is the order of ss' in W). Let H' be the $\mathcal{A}'$-subalgebra of H generated by T_s ($s \in S$); note that $H = \mathcal{A}\otimes_{\mathcal{A}'} H'$. Let $H_K = K \otimes_{\mathcal{A}} H$, $H_{K'} = K' \otimes_{\mathcal{A}'} H'$ so that $H_K = K \otimes_{K'} H_{K'}$. It is known [7] (see also 1.2 below) that the algebra H_K is canonically isomorphic to the

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group algebra $K[W]$. Hence any $E \in \mathrm{Irr}W$ can be viewed as a simple H_K-module E_v. We say that E is *ordinary* if E_v is obtained by extension of scalars from an $H_{K'}$-module; otherwise, we say that E is *exceptional*. Let Irr_0W (resp. Irr_1W) be the set of all $E \in \mathrm{Irr}W$ which are ordinary (resp. exceptional).

We define a subset $\mathcal{E}W$ of $\mathrm{Irr}W$ as follows. If W is not of type E_7, E_8, H_3, H_4, we set $\mathcal{E}W = \emptyset$. If W is of type E_7, E_8, H_3, H_4, then $\mathcal{E}W$ consists of 2^a representations of dimension 2^b where $2^a = 2$ for E_7, H_3, $2^a = 4$ for E_8, H_4 and 2^{a+b} is the largest power of 2 that divides the order of W; thus 2^b is 512, 4096, 4, 16 respectively.

When W is crystallographic we have $\mathrm{Irr}W - \mathcal{E}W \subset \mathrm{Irr}_0W$ (see [2]) and $\mathcal{E}W \subset \mathrm{Irr}_1W$ (a result of Springer); hence $\mathrm{Irr}W - \mathcal{E}W = \mathrm{Irr}_0W$ and $\mathcal{E}W = \mathrm{Irr}_1W$. The same holds when W is not crystallographic. (The fact $\mathcal{E}W \subset \mathrm{Irr}_1W$ for W of type H_3 was pointed out in [7]. The fact that any $E \in \mathrm{Irr}W - \mathcal{E}W$ is ordinary for W of type H_4 can be seen from the fact that, according to [1], E can be realized by a W-graph which is even (in the sense that the vertices can be partitioned into two subsets so that no edge connects vertices in the same subset).)

In this paper we try to understand various consequences in representation theory of the existence of exceptional representations.

1.2. Let $\{c_w; w \in W\}$ be the basis of H which in [5] was denoted by $\{C'_w; w \in W\}$. Let $\le_{LR}$, $\le_L$ be the preorders on W defined in [5] and let $\sim_{LR}$, $\sim_L$ be the corresponding equivalence relations on W (the equivalence classes are called the two-sided cells and left cells respectively). For $x, y \in W$ we write $c_xc_y = \sum_{z\in W} h_{x,y,z}c_z$. For $z \in W$ there is a unique number $a(z) \in \mathbf{N}$ such that for any x, y in W we have $h_{x,y,z} = \gamma_{x,y,z^{-1}}v^{a(z)} \mod v^{a(z)-1}\mathbf{Z}[v^{-1}]$ where $\gamma_{x,y,z^{-1}} \in \mathbf{N}$ and $\gamma_{x,y,z^{-1}} > 0$ for some x, y in W. Moreover, $z \mapsto a(z)$ is constant on any two-sided cell. (See [11].) Let J be the $\mathbf{C}$-vector space with basis $\{t_w; w \in W\}$. It has an associative $\mathbf{C}$-algebra structure given by $t_xt_y = \sum_{z\in W} \gamma_{x,y,z^{-1}}t_z$; it has a unit element of the form $\sum_{d\in\mathcal{D}} t_d$ where $\mathcal{D}$ is a subset of W consisting of certain involutions (that is elements with square 1). (See [11].) Let $h \mapsto h^\dagger$ be the algebra automorphism of H such that $T_s^da = -T_s{}^{-1}$ for $s \in S$. Now the $\mathcal{A}$-linear map $H \to \mathcal{A} \otimes J$ given by $c_x^\dagger \mapsto \sum_{d\in\mathcal{D},z\in W,d\sim_L z} h_{x,d,z}t_z$ induces an algebra isomorphism $H_K \to K \otimes J$ and (by specializing $v = 1$) an algebra isomorphism $\mathbf{C}[W] \to J$ hence an algebra isomorphism $K[W] \to K \otimes J$. (See [11].) Now if $E \in \mathrm{Irr}W$ then E_v in 1.1 is obtained as follows. We first view $K \otimes E$ as a $K \otimes J$-module E_∞ via the isomorphism $K[W] \to K \otimes J$ above and then view E_∞ as an H_K-module E_v via the isomorphism $H_K \to K \otimes J$. Note that for $x \in W$ we have

(a) $$\mathrm{tr}(c_x^\dagger, E_v) = \sum_{d\in\mathcal{D},z\in W;d\sim_L z} h_{x,d,z}\mathrm{tr}(t_z, E_\infty).$$

We show:

(b) *If $x \in W$ satisfies $x^2 = 1$, or more generally, if $x \sim_L x^{-1}$ then there exists $E \in \mathrm{Irr}W$ such that* $\mathrm{tr}(t_x, E_\infty) \neq 0$.

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