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Brauer's height zero conjecture for quasi-simple groups ☆

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ABSTRACT

We show that Brauer's height zero conjecture holds for blocks of finite quasi-simple groups. This result is used in Navarro–Späth's reduction of this conjecture for general groups to the inductive Alperin–McKay condition for simple groups.

1. Introduction

In this paper we verify that the open direction of Richard Brauer's 1955 height zero conjecture (BHZ) holds for blocks of finite quasi-simple groups:

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Main Theorem. *Let S be a finite quasi-simple group, ℓ a prime and B an ℓ-block of S. Then B has abelian defect groups if and only if all $\chi \in \mathrm{Irr}(B)$ have height zero.*

The proof of one direction of Brauer's height zero conjecture, that blocks with abelian defect groups only contain characters of height zero, was completed in [15]. Subsequently it was shown by Gabriel Navarro and Britta Späth [22] that the other direction of (BHZ) can be reduced to proving the following for all finite quasi-simple groups S:

(1) (BHZ) holds for S, and
(2) the inductive form of the Alperin–McKay conjecture holds for $S/Z(S)$.

Here, we show that the first statement holds. The main case, when S is quasi-simple of Lie type, is treated in Section 2, and then the proof of the Main Theorem is completed in Section 3.

2. Brauer's height zero conjecture for groups of Lie type

In this section we show that (BHZ) holds for quasi-simple groups of Lie type. This constitutes the central part of the proof of our Main Theorem.

Throughout, we work with the following setting. We let $\mathbf{G}$ be a connected reductive linear algebraic group over an algebraic closure of a finite field of characteristic p, and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism with finite group of fixed points $\mathbf{G}^F$. It is well-known that apart from finitely many exceptions, all finite quasi-simple groups of Lie type can be obtained as $\mathbf{G}^F/Z$ for some central subgroup $Z \le \mathbf{G}^F$ by choosing $\mathbf{G}$ simple of simply connected type.

We let $\mathbf{G}^*$ be dual to $\mathbf{G}$, with compatible Steinberg endomorphism again denoted F. Recall that by the results of Lusztig the set $\mathrm{Irr}(\mathbf{G}^F)$ of complex irreducible characters of $\mathbf{G}^F$ is a disjoint union of rational Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$, where s runs over the semisimple elements of $\mathbf{G}^{*F}$ up to conjugation.

2.1. Groups of Lie type in their defining characteristic

We first consider the easier case of groups of Lie type in their defining characteristic, where we need the following:

Lemma 2.1. *Let $\mathbf{G}$ be simple, not of type A_1, with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then every coset of $[\mathbf{G}^F, \mathbf{G}^F]$ in $\mathbf{G}^F$ contains a (semisimple) element centralising a root subgroup of $\mathbf{G}^F$.*

Proof. First note that by inspection any of the rank 2 groups $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, and $\mathrm{S}_4(q)$ (and hence also $\mathrm{U}_4(q)$) contains a root subgroup $U \cong \mathbb{F}_q^+$ all of whose non-identity elements are conjugate under a maximally split torus. Now if $\mathbf{G}$ is not of type A_1 with

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