ARTICLE IN PRESS YJABR:15824

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

$U_n(q)$ acting on flags and supercharacters [☆]

Richard Dipper [a,*], Qiong Guo [b,*]

[a] *Institut für Algebra und Zahlentheorie, Universität Stuttgart, 70569 Stuttgart, Germany*
[b] *College of Sciences, Shanghai Institute of Technology, 201418 Shanghai, PR China*

ARTICLE INFO

Article history:
Received 10 November 2015
Available online xxxx
Communicated by B. Srinivasan, M. Collins and G. Lehrer

Dedicated to the memory of J.A. Green

MSC:
primary 20C15, 20D15
secondary 20C33, 20D20

Keywords:
Flags
Monomial linearization
Supercharacter

ABSTRACT

Let $U = U_n(q)$ be the group of lower unitriangular $n \times n$-matrices with entries in the field $\mathbb{F}_q$ with q elements for some prime power q and $n \in \mathbb{N}$. We investigate the restriction to U of the permutation action of $GL_n(q)$ on flags in the natural $GL_n(q)$-module $\mathbb{F}_q^n$. Applying our results to the special case of flags of length two we obtain a complete decomposition of the permutation representation of $GL_n(q)$ on the cosets of maximal parabolic subgroups into irreducible $\mathbb{C}U$-modules.

1. Introduction

The irreducible complex characters of the finite general linear group $GL_n(q)$ have been determined by J.A. Green in his landmark paper [8]. They subdivide naturally

☆ This work was partially supported by the DFG priority programme SPP 1388 in representation theory, no. 99028426.
* Corresponding authors.
E-mail addresses: richard.dipper@mathematik.uni-stuttgart.de (R. Dipper), qiongguo@sit.edu.cn (Q. Guo).

http://dx.doi.org/10.1016/j.jalgebra.2016.07.014

Please cite this article in press as: R. Dipper, Q. Guo, $U_n(q)$ acting on flags and supercharacters, J. Algebra (2016), http://dx.doi.org/10.1016/j.jalgebra.2016.07.014

into families, called Harish-Chandra series, labeled by conjugacy classes of semisimple elements of $GL_n(q)$. In this paper we are mainly concerned with the series attached to the identity element of $GL_n(q)$. Those are afforded precisely by the irreducible constituents of the permutation representation of $GL_n(q)$ acting on the cosets of a Borel subgroup B, which are called unipotent Specht modules. Many aspects of the representation theory of $GL_n(q)$ may be described analogously to the representation theory of the symmetric group $\mathfrak{S}_n$ – indeed one expects the theory of $GL_n(q)$ to translate into that of $\mathfrak{S}_n$ by setting q equal to 1.

It is a classical result that the Specht module S^λ for $\mathfrak{S}_n$, where λ is a partition of n, is integrally defined. It comes with a certain distinguished integral basis, called standard basis, which is labeled by standard λ-tableaux. This suggests that a q-version of this should hold for unipotent Specht modules $S(\lambda)$ for $GL_n(q)$. More precisely, to each standard λ-tableau $\mathfrak{s}$, there should be attached a polynomial $g_\mathfrak{s}(x) \in \mathbb{Z}[x]$ with $g_\mathfrak{s}(1) = 1$ and $g_\mathfrak{s}(q)$ many elements of $S(\lambda)$ such that all these elements form a basis of $S(\lambda)$, where $\mathfrak{s}$ runs through all the standard λ-tableaux. In [3,4,7,9] such standard bases for $S(\lambda)$ were constructed in the special case of 2-part partitions λ.

One way to define unipotent Specht modules for $GL_n(q)$ in a characteristic free way is provided by James' kernel intersection theorem [10, Theorem 15.19]: For a partition λ of n, the unipotent Specht module $S(\lambda)$ for $GL_n(q)$ over any field K of characteristic not dividing q is defined as intersection of the kernels of all homomorphisms $\phi : M(\lambda) \to M(\mu)$ for all partition μ of n dominating λ. Here $M(\lambda)$ denotes the permutation representation of $GL_n(q)$ acting by right translation on the set of all λ-flags in the natural $GL_n(q)$-module $V = \mathbb{F}_q^n$, where $\mathbb{F}_q$ is the field with q elements. In a recent paper, Andrews [13] gave an alternate construction of the unipotent Specht modules based on generalized Gelfand–Graev characters. If $K = \mathbb{C}$ then $S(\lambda)$ are irreducible unipotent $GL_n(q)$-modules appearing as irreducible constituents of the permutation module of $GL_n(q)$ acting on flags in V.

An important ingredient in [9] consists of an analysis of the restriction of $M(\lambda)$ to the unitriangular group $U = U_n(q)$ of $GL_n(q)$. The group algebra KU is semisimple for all fields K of characteristic not dividing q. In [9] decomposing the restriction of $M(\lambda)$ to U completely into irreducible U-modules for 2-part partitions λ, and then applying James' kernel intersection theorem, the main results of [4] were reproved, giving the standard basis of $S(\lambda)$ a representation theoretic interpretation. This new approach in [9] bears resemblance to Kirillov's orbit method [12] and the supercharacter theory of U introduced by André [1] and Yan [14]. Thus making this remarkable connection precise and generalize it to arbitrary compositions λ of n is hoped to be a first step towards constructing standard bases of unipotent Specht modules. This is the main goal of this paper.

The permutation modules $M(\lambda)$ decompose as U-modules into direct sums of $\mathfrak{s}$-components $M_\mathfrak{s}$, where $\mathfrak{s}$ runs through the set $\mathrm{RStd}(\lambda)$ of row standard λ-tableaux. If, for $\mathfrak{s} \in \mathrm{RStd}(\lambda)$, $d = d(\mathfrak{s}) \in \mathfrak{S}_n$ denotes the permutation taking the initial λ-tableau to $\mathfrak{s}$, the intersection $U^d \cap U$ is a semidirect product $U_K = U_J \rtimes U_L$ of pattern subgroups.