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Anchors of irreducible characters

Radha Kessar [a], Burkhard Külshammer [b,*,1], Markus Linckelmann [a]

[a] Department of Mathematics, City University London, EC1V 0HB, United Kingdom
[b] Institut für Mathematik, Friedrich-Schiller-Universität, 07743 Jena, Germany

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ABSTRACT

Given a prime number p, every irreducible character χ of a finite group G determines a unique conjugacy class of p-subgroups of G which we will call the *anchors* of χ. This invariant has been considered by Barker in the context of finite p-solvable groups. Besides proving some basic properties of these anchors, we investigate the relation to other p-groups which can be attached to irreducible characters, such as defect groups, vertices in the sense of J.A. Green and vertices in the sense of G. Navarro.

1. Introduction

Let p be a prime number and $\mathcal{O}$ a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p and field of fractions K of characteristic 0. For G a finite group, we denote by $\mathrm{Irr}(G)$ the set of characters of the simple KG-modules.

* Corresponding author.
E-mail addresses: radha.kessar.1@city.ac.uk (R. Kessar), kuelshammer@uni-jena.de (B. Külshammer), markus.linckelmann@city.ac.uk (M. Linckelmann).

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For $\chi \in \mathrm{Irr}(G)$, we denote by e_χ the unique primitive idempotent in $Z(KG)$ satisfying $\chi(e_\chi) \neq 0$. The $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ in the simple K-algebra KGe_χ is a G-interior $\mathcal{O}$-algebra, via the group homomorphism $G \to (\mathcal{O}Ge_\chi)^\times$ sending $g \in G$ to ge_χ. Since $(\mathcal{O}Ge_\chi)^G = Z(\mathcal{O}Ge_\chi)$ is a subring of the field $Z(KGe_\chi)$, it follows that $\mathcal{O}Ge_\chi$ is a primitive G-interior $\mathcal{O}$-algebra. In particular, $\mathcal{O}Ge_\chi$ is a primitive G-algebra. By the fundamental work of J.A. Green [7], it has a defect group. This is used in work of Barker [1] to prove a part of a conjecture of Robinson (cf. [24, 4.1, 5.1]) for blocks of finite p-solvable groups. In order to distinguish this invariant from defect groups of blocks and from vertices of modules, we introduce the following terminology.

Definition 1.1. Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. An *anchor of* χ is a defect group of the primitive G-interior $\mathcal{O}$-algebra $\mathcal{O}Ge_\chi$.

By the definition of defect groups, an anchor of an irreducible character χ of G is a subgroup P of G which is minimal with respect to $e_\chi \in (\mathcal{O}Ge_\chi)^G_P$, where $(\mathcal{O}Ge_\chi)^G_P$ denotes the image of the relative trace map $\mathrm{Tr}^G_P : (\mathcal{O}Ge_\chi)^P \to (\mathcal{O}Ge_\chi)^G$. Green's general theory in [7, §5] implies that the anchors of χ form a conjugacy class of p-subgroups of G.

For the remainder of the paper we make the blanket assumption that K and k are splitting fields for the finite groups arising in the statements below. In a few places, this assumption is not necessary; see the Remark 1.7 below.

Theorem 1.2. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Let B be the block of $\mathcal{O}G$ containing χ and let L be an $\mathcal{O}G$-lattice affording χ. Let P be an anchor of χ and denote by ΔP the image $\{(x,x) \,|\, x \in P\}$ of P under the diagonal embedding of G in $G \times G$. The following hold.*

(a) *P is contained in a defect group of B.*
(b) *P contains a vertex of L.*
(c) *We have $O_p(G) \leq P$.*
(d) *The suborder $\mathcal{O}Pe_\chi$ of $\mathcal{O}Ge_\chi$ is local, and $\mathcal{O}Ge_\chi$ is a separable extension of $\mathcal{O}Pe_\chi$.*
(e) *ΔP is contained in a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$ and $P \times P$ contains a vertex of $\mathcal{O}Ge_\chi$. Moreover, ΔP is a vertex of $\mathcal{O}Ge_\chi$ if and only if χ is of defect zero.*

For G a finite group, we denote by $\mathrm{IBr}(G)$ the set of $\mathcal{O}$-valued Brauer characters of the simple kG-modules. We denote by G° the set of p'-elements in G, and for χ a K-valued class function on G, we denote by χ° the restriction of χ to G°.

Theorem 1.3. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let B be the block of $\mathcal{O}G$ containing χ and let L be an $\mathcal{O}G$-lattice affording χ. Let P be an anchor of χ. The following hold.*

(a) *If $\chi^\circ \in \mathrm{IBr}(G)$, then L is unique up to isomorphism, P is a vertex of L, and $P \times P$ is a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$.*