ARTICLE IN PRESS YJABR:15931

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Hall polynomials for tame type ☆

Bangming Deng, Shiquan Ruan *

Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

A R T I C L E I N F O

Article history:
Received 30 November 2015
Available online xxxx
Communicated by B. Srinivasan, M. Collins and G. Lehrer

To the memory of Professor J.A. Green

MSC:
14F05
16G20
17B37

Keywords:
Weighted projective line
Hall polynomial
Ringel–Hall algebra
Green's formula

A B S T R A C T

In the present paper we prove that Hall polynomial exists for each triple of decomposition sequences which parameterize isomorphism classes of coherent sheaves of a domestic weighted projective line $\mathbb{X}$ over finite fields. These polynomials are then used to define the generic Ringel–Hall algebra of $\mathbb{X}$ as well as its Drinfeld double. Combining this construction with a result of Cramer, we show that Hall polynomials exist for tame quivers, which not only refines a result of Hubery, but also confirms a conjecture of Berenstein and Greenstein.

1. Introduction

Inspired by the work of Steinitz [29] and Hall [12], Ringel [19,20] introduced the Hall algebra $H(A)$ of a finite dimensional algebra A, whose structure constants are given by the so-called Hall numbers, and proved that if A is hereditary and representation finite,

☆ Supported partially by the Natural Science Foundation of China (Grant Nos. 11271043, 11331006).

* Corresponding author.
E-mail addresses: bmdeng@math.tsinghua.edu.cn (B. Deng), sqruan@math.tsinghua.edu.cn (S. Ruan).

http://dx.doi.org/10.1016/j.jalgebra.2016.10.002

then $H(A)$ is isomorphic to the positive part of the corresponding quantized enveloping algebra. By introducing a bialgebra structure on $H(A)$, Green [10] then generalized Ringel's work to arbitrary finite dimensional hereditary algebra A and showed that the composition subalgebra of $H(A)$ generalized by simple A-modules gives a realization of the positive part of the quantized enveloping algebra associated with A. The proof of the compatibility of multiplication and comultiplication on $H(A)$ is based on a marvelous formula arising from the homological properties of A-modules, called Green's formula. We remark that Lusztig [15] has obtained a geometric construction of quantized enveloping algebras in terms of perverse sheaves on representation varieties of quivers.

In case A is representation finite and hereditary, Ringel [19] showed that the structure constants of $H(A)$ are actually integer polynomials in the cardinalities of finite fields. The proof is based on a basic property of the module category of A, namely, the directedness of its Auslander–Reiten quiver. These polynomials are called Hall polynomials as in the classical case; see [16]. Then one can define the generic Hall algebra $H_{\boldsymbol{q}}(A)$ over the polynomial ring $\mathbb{Q}[\boldsymbol{q}]$ and its degeneration $H_1(A)$ at $\boldsymbol{q} = 1$. It was shown by Ringel [22] that $H_1(A)$ is isomorphic to the positive part of the universal enveloping algebra of the semisimple Lie algebra associated with A. Since then, much subsequent work was devoted to the study of Hall polynomials for various classes of algebras. Ringel [21] has calculated Hall polynomials for three indecomposable modules over a representation-finite hereditary algebra. Some Hall polynomials for representations over the Kronecker quiver has been calculated in [31,25]. Recently, Hubery [13] provided an elegant proof of the existence of Hall polynomials for all Dynkin and cyclic quivers by an inductive argument based on Green's formula mentioned above. Moreover, he proved that Hall polynomials exist for all tame (affine) quivers with respect to the decomposition classes of Bongartz and Dudek [1].

Inspired by the work of Hubery [13], the main purpose of the present paper is to study Hall polynomials for coherent sheaves of a domestic weighted projective line $\mathbb{X}$ over finite fields. The key idea is again the use of Green's formula. More precisely, we extend the notion of decomposition classes to that of decomposition sequences, which parameterize isoclasses (isomorphism classes) of coherent sheaves of $\mathbb{X}$ over finite fields, and show that Hall polynomial exists for each triple of decomposition sequences. These polynomials are then applied to define an algebra $H_{\boldsymbol{v}}(\mathbb{X})$ which is a free module over the Laurent polynomial ring $\mathbb{Q}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ with a basis all the decomposition sequences. By extending $H_{\boldsymbol{v}}(\mathbb{X})$ via formally adding certain elements constructed in [3], we obtain the generic Ringel–Hall algebra $\boldsymbol{\mathcal{H}}_{\boldsymbol{v}}(\mathbb{X})$ of $\mathbb{X}$. By further introducing Green's pairing on $\boldsymbol{\mathcal{H}}_{\boldsymbol{v}}(\mathbb{X})$, we construct its Drinfeld double $D\boldsymbol{\mathcal{H}}_{\boldsymbol{v}}(\mathbb{X})$ over $\mathbb{Q}(\boldsymbol{v})$. Combining this construction with [5, Prop. 5], we show that Hall polynomials exist for decomposition sequences associated with a tame quiver. This result refines the main theorem of Hubery [13] and also confirms a conjecture of Berenstein and Greenstein [2, Conj. 3.4].

The paper is organized as follows. Section 2 gives a brief introduction on the category of coherent sheaves over a weighted projective line $\mathbb{X}$ and recalls the definition of the Ringel–Hall algebra of $\mathbb{X}$ over a finite field and the Green's formula as well. In Section 3

Please cite this article in press as: B. Deng, S. Ruan, Hall polynomials for tame type, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.10.002