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Algebras with non-periodic bounded modules

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ABSTRACT

We study weakly symmetric special biserial algebra of infinite representation type. We show that usually some socle deformation of such an algebra has non-periodic bounded modules. The exceptions are precisely the algebras whose Brauer graph is a tree with no multiple edges. If the algebra has a non-periodic bounded module then its Hochschild cohomology cannot satisfy the finite generation property (Fg) introduced in [9].

1. Introduction

Assume Λ is a finite-dimensional selfinjective algebra over some field K. If M is a finite-dimensional non-projective Λ-module, let

$$\ldots \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

be a minimal projective resolution of M. The module M is called bounded if the dimensions of the projectives P_n have a common upper bound, that is, M has complexity one. The kernel of d_n is the syzygy $\Omega^n(M)$; we say that the module M is periodic if $\Omega^d(M) \cong M$ for some $d \geq 1$. A periodic module has complexity one but the converse

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need not hold. We call a module M a *criminal* if it has complexity one but is not periodic. We would like to understand which algebras have criminals.

J. Alperin proved in [1] that the group algebra of a finite group does not have criminals when the coefficient field is algebraic over its prime field. On the other hand, R. Schulz discovered that there are four-dimensional selfinjective algebras which have criminals, see [16]. In the context of commutative algebra, there is a similar problem. Eisenbud proved in [6] that for complete intersections, if a finitely generated module has bounded Betti numbers then it is eventually periodic. He conjectured that this should be true for any commutative Noetherian local ring. However, counterexamples were constructed by Gasharov and Peeva [13].

Subsequently, a theory of support varieties was developed for modules of group algebras of finite groups. This is based on group cohomology and depends crucially on that the fact that it is Noetherian. It follows from this theory that a group algebra over an arbitrary field does not have criminals, so that Alperin's theorem holds in general; for a proof see 2.24.4 in [3]. More recently, a support variety theory was developed for modules of selfinjective algebras, based on Hochschild cohomology [17,9]. This also requires suitable finite generation, namely the Hochschild cohomology $HH^*(\Lambda)$ should be Noetherian and the ext-algebra of Λ should be finitely generated as a module over $HH^*(\Lambda)$. This condition is called (Fg) in [18], it is equivalent to (Fg1, 2) in [9]. Again, if (Fg) holds for Λ, so that Λ-modules have support varieties, then Λ does not have criminals (see 5.3 in [9]).

The algebras studied by Schulz therefore do not satisfy (Fg). More generally, weakly symmetric algebra with radical cube zero were investigated in [10,11]. The algebras in these papers which have criminals happen to be special biserial, therefore one may ask when a special biserial weakly symmetric algebra has criminals. Of course, if an algebra has a chance to have criminals it must have infinite representation type.

Here we study special biserial weakly symmetric K-algebras of infinite representation type, we assume K is an algebraically closed field such that there exists $x \in K$ with $x^m \neq 1$ for all m a natural number. This condition is crucial; we will construct modules of complexity one and whose period depends on the multiplicative order of a non-zero scalar (see Proposition 3.7). An algebra is special biserial weakly symmetric if its basic algebra satisfies 2.1. Existence of criminals is invariant under Morita equivalence, and we will work throughout with basic algebras. We assume that the algebra is indecomposable, so that its quiver is connected. The algebras in 2.1 have socle relations involving scalar parameters, so that we have a family of algebras, which we write as $\Lambda_{\mathbf{q}}$ where $\mathbf{q}$ is the collection of the socle scalars. Each algebra in a family is a socle deformation of the algebra $\Lambda_{\mathbf{1}}$ for which all socle scalars are equal to 1.

Recall that if Λ and Γ are selfinjective, then Γ is a socle deformation of Λ if $\Gamma/\mathrm{soc}(\Gamma)$ is isomorphic to $\Lambda/\mathrm{soc}(\Lambda)$. For example when the field has characteristic 2 then the algebra studied by Schulz is a socle deformation of the group algebra of a Klein 4-group. There are similar socle deformations for group algebras of dihedral 2-groups, which are also special biserial and weakly symmetric.