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Representations of quivers over the algebra of dual numbers

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A B S T R A C T

The representations of a quiver Q over a field k (the kQ-modules, where kQ is the path algebra of Q over k) have been studied for a long time, and one knows quite well the structure of the module category $\operatorname{mod} kQ$. It seems to be worthwhile to consider also representations of Q over arbitrary finite-dimensional k-algebras A. Here we draw the attention to the case when $A = k[\epsilon]$ is the algebra of dual numbers (the factor algebra of the polynomial ring $k[T]$ in one variable T modulo the ideal generated by T^2), thus to the Λ-modules, where $\Lambda = kQ[\epsilon] = kQ[T]/\langle T^2\rangle$. The algebra Λ is a 1-Gorenstein algebra, thus the torsionless Λ-modules are known to be of special interest (as the Gorenstein-projective or maximal Cohen–Macaulay modules). They form a Frobenius category $\mathcal{L}$, thus the corresponding stable category $\underline{\mathcal{L}}$ is a triangulated category. As we will see, the category $\mathcal{L}$ is the category of perfect differential kQ-modules and $\underline{\mathcal{L}}$ is the corresponding homotopy category. The category $\underline{\mathcal{L}}$ is triangle equivalent to the orbit category of the derived category $D^b(\operatorname{mod} kQ)$ modulo the shift and the homology functor $H\colon \operatorname{mod}\Lambda \to \operatorname{mod} kQ$ yields a bijection between the indecomposables in $\underline{\mathcal{L}}$ and those in $\operatorname{mod} kQ$. Our main interest lies in the inverse, it is given by the minimal $\mathcal{L}$-approximation. Also, we will determine the kernel of the restriction of the functor H to $\mathcal{L}$ and describe the Auslander–Reiten quivers of $\mathcal{L}$ and $\underline{\mathcal{L}}$.

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Relationship between abelian categories and triangulated categories
Homology functor
Ghost maps
Auslander–Reiten quiver of Gorenstein-projective modules
Covering theory
Graded group algebras

Introduction

Throughout the paper, k will be a field and Q will be a finite connected acyclic quiver. The starting point for the considerations of this paper is the following result which concerns the structure of the homotopy category of perfect differential kQ-modules. This assertion should be well-known, but we could not find a reference.

Let us recall that given a ring R, a differential R-module is by definition a pair (N, ϵ) where N is an R-module and ϵ an endomorphism of N such that $\epsilon^2 = 0$. If (N, ϵ) and (N', ϵ') are differential R-modules, a morphism $f: (N, \epsilon) \to (N', \epsilon')$ is given by an R-linear map $f: N \to N'$ such that $\epsilon' f = f\epsilon$. The morphism $f: (N, \epsilon) \to (N', \epsilon')$ is said to be *homotopic to zero* provided there exists an R-linear map $h: N \to N'$ such that $f = h\epsilon + \epsilon' h$. A differential R-module (N, ϵ) is said to be *perfect* provided N is a finitely generated projective R-module. We denote by $\operatorname{diff}_{\mathrm{perf}}(R)$ the category of perfect differential R-modules, and by $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ the corresponding homotopy category. Let us denote by H the homology functor: it attaches to a differential R-module (N, ϵ) the R-module $H(N, \epsilon) = \operatorname{Ker} \epsilon / \operatorname{Im} \epsilon$. It is well-known that H vanishes on the maps which are homotopic to zero.

If R is noetherian, let us denote by $D^b(\operatorname{mod} R)$ the bounded derived category of finitely generated R-modules. This is a triangulated category and its shift functor will be denoted by [1].

Theorem 1. (a) *The category* $\operatorname{diff}_{\mathrm{perf}}(kQ)$ *of perfect differential* kQ*-module is a Frobenius category. The corresponding stable category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ *is the homotopy category of perfect differential* kQ*-modules. This category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ *is the orbit category* $D^b(\operatorname{mod} kQ)/[1]$.

(b) *The homology functor* $H: \operatorname{diff}_{\mathrm{perf}}(kQ) \to \operatorname{mod} kQ$ *is a full and dense functor which furnishes a bijection between the indecomposables in the homotopy category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(kQ)$ *and those in* $\operatorname{mod} kQ$. *It yields a quiver embedding* ι *of the Auslander–Reiten quiver of* $\operatorname{mod} kQ$ *into the Auslander–Reiten quiver of the homotopy category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(kQ)$.

We should remark that the study of differential modules themselves may have been neglected by the algebraists, however it is clear that the graded version, namely complexes, play an important role in many parts of mathematics. Theorem 1 is an immediate consequence of well-known results concerning perfect complexes over kQ: the category of perfect complexes is a Frobenius category, thus the corresponding stable category

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