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On the modular composition factors of the Steinberg representation

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A R T I C L E I N F O

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A B S T R A C T

Let G be a finite group of Lie type and St_k be the Steinberg representation of G, defined over a field k. We are interested in the case where k has prime characteristic ℓ and St_k is reducible. Tinberg has shown that the socle of St_k is always simple. We give a new proof of this result in terms of the Hecke algebra of G with respect to a Borel subgroup and show how to identify the simple socle of St_k among the principal series representations of G. Furthermore, we determine the composition length of St_k when $G = \mathrm{GL}_n(q)$ or G is a finite classical group and ℓ is a so-called linear prime.

1. Introduction

Let G be a finite group of Lie type and St_k be the Steinberg representation of G, defined over a field k. Steinberg [31] showed that St_k is irreducible if and only if $[G : B]1_k \neq 0$ where B is a Borel subgroup of G. We shall be concerned here with the case where St_k is reducible. There is only very little general knowledge about the structure

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of St_k in this case. We mention the works of Tinberg [34] (on the socle of St_k), Hiss [19] and Khammash [27] (on trivial composition factors of St_k) and Gow [15] (on the Jantzen filtration of St_k).

One of the most important open questions in this respect seems to be to find a suitable bound on the length of a composition series of St_k. Typically, this problem is related to quite subtle information about decomposition numbers; see, for example, Landrock–Michler [28] and Okuyama–Waki [30] where this is solved for groups with a BN-pair of rank 1. For groups of larger BN-rank, this problem is completely open.

In this paper, we discuss two aspects of this problem.

Firstly, Tinberg [34] has shown that the socle of St_k is always simple, using results of Green [16] applied to the endomorphism algebra of the permutation module $k[G/U]$ where U is a maximal unipotent subgroup. After some preparations in Sections 2, we show in Section 3 that a similar argument works with U replaced by B. Since the corresponding endomorphism algebra (or "Hecke algebra") is much easier to describe and its representation theory is quite well understood, this provides new additional information. For example, if $G = \mathrm{GL}_n(q)$, then we can identify the partition of n which labels the socle of St_k in James' [24] parametrisation of the unipotent simple modules of G; see Example 3.6. Quite remarkably, this involves a particular case of the "Mullineux involution" — and an analogue of this involution for other types of groups.

In another direction, we consider the partition of the simple kG-modules into Harish–Chandra series, as defined by Hiss [20]. Dipper and Gruber [6] have developed a quite general framework for this purpose, in terms of so-called "projective restriction systems". In Section 4, we shall present a simplified, self-contained version of parts of this framework which is tailored towards applications to St_k. This yields, first of all, new proofs of some of the results of Szechtman [33] on St_k for $G = \mathrm{GL}_n(q)$; moreover, in Example 4.9, we obtain an explicit formula for the composition length of St_k in this case. Analogous results are derived for groups of classical type in the so-called "linear prime" case, based on [10,17,18]. For example, St_k is seen to be multiplicity-free with a unique simple quotient in these cases — properties which do not hold in general for non-linear primes.

2. The Steinberg module and the Hecke algebra

Let G be a finite group and $B, N \subseteq G$ be subgroups which satisfy the axioms for an "algebraic group with a split BN-pair" in [2, §2.5]. We just recall explicitly those properties of G which will be important for us in the sequel. Firstly, there is a prime number p such that we have a semidirect product decomposition $B = U \rtimes H$ where $H = B \cap N$ is an abelian group of order prime to p and U is a normal p-subgroup of B. The group H is normal in N and $W = N/H$ is a finite Coxeter group with a canonically defined generating set S; let $l\colon W \to \mathbb{N}_0$ be the corresponding length function. For each $w \in W$, let $n_w \in N$ be such that $Hn_w = w$. Then we have the Bruhat decomposition