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On the number of simple modules in a block of a finite group

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ABSTRACT

We prove that if B is a p-block with non-trivial defect group D of a finite p-solvable group G, then $\ell(B) < p^r$, where r is the sectional rank of D. We remark that there are infinitely many p-blocks B with non-Abelian defect groups and $\ell(B) = p^r - 1$. We conjecture that the inequality $\ell(B) \leq p^r$ holds for an arbitrary p-block with defect group of sectional rank r. We show this to hold for a large class of p-blocks of various families of quasi-simple and nearly simple groups.

1. Introduction

Almost 60 years ago, Richard Brauer posed his fundamental problem on $k(B)$, the number of ordinary irreducible characters in a p-block B of defect d of a finite group: Is it always the case that $k(B) \leq p^d$? Despite partial progress, this question remains open

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to the present day. Here, we propose a conjecture of a similar flavour for the number of irreducible Brauer characters in a p-block. In order to formulate it, we need the following notion:

For a p-block B let $s(B)$ denote the sectional p-rank of a defect group D of B, that is, the largest rank of an elementary Abelian section of D. Further, let $\ell(B)$ denote the number of isomorphism classes of simple modules in B.

Conjecture 1. *Let B be a p-block of a finite group. Then $\ell(B) \leq p^{s(B)}$.*

Let us mention that in all of our results, the inequality in Conjecture 1 turns out to be strict for blocks of positive defect, unless possibly when $p = 3$ and B has extra-special defect group of order 27, see Remark 5.4.

The first main result of this paper is the proof of our conjecture (with strict inequality) in the case of p-solvable groups, which will be given in Section 2 using the positive solution of the $k(GV)$-problem:

Theorem 2. *Let B be a p-block of a finite p-solvable group with non-zero defect. Then $\ell(B) < p^{s(B)}$.*

This has the following consequence:

Corollary 3. *Let B be a block with an abelian defect group, and assume that B either satisfies Alperin's weight conjecture, or Broué's perfect isometry conjecture. Then Conjecture 1 holds for B, with strict inequality if B has non-zero defect.*

Our second main result concerns certain blocks of nearly simple groups. More precisely we show the following:

Theorem 4. *Conjecture 1 holds for the following p-blocks of nearly simple groups:*

(a) *all p-blocks of all covering groups of symmetric, alternating, sporadic, special linear and special unitary groups;*
(b) *all p-blocks of all quasi-simple groups of Lie type in characteristic p;*
(c) *all unipotent p-blocks of quasi-simple groups G of Lie type in characteristic different from p when p is good for G; and*
(d) *the principal p-blocks of all quasi-simple groups of Lie type.*

This is shown in Propositions 5.2, 5.3, 6.1, Theorems 6.6, 6.7 and Proposition 6.10 respectively. The proofs in those cases involve interesting combinatorial questions on numbers of unipotent characters and of unipotent classes in groups of Lie type.

In view of Conjecture 1 it would be interesting to know whether or under what conditions the sectional rank of a defect group is an invariant under Morita equivalences of blocks.

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