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Basic superranks for varieties of algebras

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ABSTRACT

We introduce the notion of basic superrank for varieties of algebras generalizing the notion of basic rank. First we consider a number of varieties of nearly associative algebras over a field of characteristic 0 that have infinite basic ranks and calculate their basic superranks which turns out to be finite. Namely we prove that the variety of alternative metabelian (solvable of index 2) algebras possesses the two basic superranks $(1, 1)$ and $(0, 3)$; the varieties of Jordan and Malcev metabelian algebras have the unique basic superranks $(0, 2)$ and $(1, 1)$, respectively. Furthermore, for arbitrary pair $(r, s) \neq (0, 0)$ of nonnegative integers we provide a variety that has the unique basic superrank (r, s). Finally, we construct some examples of nearly associative varieties that do not possess finite basic superranks.

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Basic superrank of variety
Basic spectrum of variety

Introduction

Throughout the paper, all algebras are considered over a field of characteristic 0. Let $\mathcal{V}$ be a variety of algebras and $\mathcal{V}_r$ be a subvariety of $\mathcal{V}$ generated by the free $\mathcal{V}$-algebra of rank r. Then one can consider the chain

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_r \subseteq \cdots \subseteq \mathcal{V},$$

where $\mathcal{V} = \bigcup_r \mathcal{V}_r$. If this chain stabilizes, then the minimal number r with the property $\mathcal{V}_r = \mathcal{V}$ is called the *basic rank* of the variety $\mathcal{V}$ and is denoted by $r_b(\mathcal{V})$ (see [13]). Otherwise, we say that $\mathcal{V}$ has the *infinite basic rank* $r_b(\mathcal{V}) = \aleph_0$.

Let us recall the main results on the basic ranks of the *varieties of associative* (Assoc), *Lie* (Lie), *alternative* (Alt), *Malcev* (Malc), and some other *algebras*. It was first shown by A.I. Mal'cev [13] that $r_b(\mathrm{Assoc}) = 2$. A.I. Shirshov [28] proved that $r_b(\mathrm{Lie}) = 2$ and $r_b(\mathrm{SJord}) = 2$, where SJord is the variety generated by all special Jordan algebras. In 1958, A.I. Shirshov posed a problem on finding basic ranks for alternative and some other varieties of nearly associative algebras [3, Problem 1.159]. In 1977, the second author proved that $r_b(\mathrm{Alt}) = r_b(\mathrm{Malc}) = \aleph_0$ [18,28]. The similar fact for the variety of algebras of type $(-1, 1)$ was established by S.V. Pchelintsev [14]. Note that the basic ranks of the varieties of Jordan and right alternative algebras are still unknown.

A proper subvariety of associative algebras can be of infinite basic rank as well. For instance, so is the variety $\mathrm{Var}\,\mathrm{G}$ generated by the Grassmann algebra G on infinite number of generators, or the variety defined by the identity $[x, y]^n = 0,\ n > 1$.

In 1986, A.R. Kemer [10,11] solved affirmatively the famous Specht problem [21] using the tool of superalgebras. Recall that a variety $\mathcal{V}$ of algebras is called *Spechtian* or is said *to have the Specht property* if every algebra of $\mathcal{V}$ possesses a finite basis for its identities. The Kemer Theorem states the Specht property of the variety of associative algebras. This result is obtained by certain reduction to the case of graded identities of finite dimensional superalgebras. Namely it is proved that the ideal of identities of arbitrary associative algebra coincides with the ideal of identities of the Grassmann envelope of some finite dimensional superalgebra.

This result suggests a generalization of the notion of basic rank. Namely we shall say that a variety $\mathcal{V}$ has a finite basic superrank if it can be generated by the Grassmann envelope of some finitely generated superalgebra. Then Kemer's result implies that every variety of associative algebras has a finite basic superrank. This means that the notion of basic superrank is a more refined one than that of basic rank, and we can distinguish varieties of infinite basic rank by their superranks.

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