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On certain recurrent and automatic sequences in finite fields

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A B S T R A C T

In this work we consider the general question: for a given algebraic formal power series with coefficients in a finite field, what kind of regularity (if any) can be expected for the partial quotients of the above power series in continued fraction expansion? Such a question is natural, since by a theorem of Christol, the coefficients of an algebraic power series over a finite field form an automatic sequence. Certain algebraic continued fractions are such that the sequence of the leading coefficients of the partial quotients is automatic. Here we give a rather general family of such sequences. Moreover, inspired by these examples, we give two criteria on automatic sequences, which allow us to obtain two new families of automatic sequences in an arbitrary finite field.

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1. Introduction

For a given algebraic power series over a finite field, by a theorem of Christol, it is well known that the coefficients of the power series in question form an automatic sequence (see Theorem 1 below). Then one can ask what kind of regularity can be expected for the partial quotients of the above power series in continued fraction expansion. This question was put forward at first by Mendès France, initiated by Allouche [1], Allouche et al. [2], and continued by Mkaouar [20] and Yao [21]. Until now there are only examples and counterexamples, but not any general result.

The present work is a continuation of our article [18] in which we have approached the question from another point of view: consider the leading coefficients of the partial quotients instead of the partial quotients themselves. To know more about the motivation and the history, the reader may consult the introduction of [18] and the references therein.

Let $\mathbb{F}_q$ be the finite field containing q elements, with $q = p^s$ where p is a prime number and s is an integer such that $s \geqslant 1$. We denote by $\mathbb{F}(q)$ the field of power series in $1/T$, with coefficients in $\mathbb{F}_q$, where T is a formal indeterminate. Hence, an element in $\mathbb{F}(q)$ can be written as $\alpha = \sum_{k \leqslant k_0} u(k)T^k$, with $k_0 \in \mathbb{Z}$ and $u(k) \in \mathbb{F}_q$ for all integers k such that $k \leqslant k_0$. These fields of power series are analogues of the field of real numbers. As in the real case, it is well known that the sequence of coefficients of this power series α, $(u(k))_{k \leqslant k_0}$, is ultimately periodic if and only if α is rational, i.e., $\alpha \in \mathbb{F}_q(T)$. Moreover and remarkably, due to the rigidity of the positive characteristic case, this sequence of coefficients, for all elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$, belongs to a class of particular sequences introduced by computer scientists. The theorem below can be found in the work [8] of Christol (see also the article [9] of Christol, Kamae, Mendès France, and Rauzy).

Theorem 1 *(Christol). Let α in $\mathbb{F}(q)$ with $q = p^s$. Let $\big(u(k)\big)_{k \leqslant k_0}$ be the sequence of digits of α and $v(n) = u(-n)$ for all integers $n \geqslant 0$. Then α is algebraic over $\mathbb{F}_q(T)$ if and only if the following set of subsequences of $(v(n))_{n \geqslant 0}$*

$$K(v) = \Big\{ \big(v(p^i n + j)\big)_{n \geqslant 0} \mid i \geqslant 0,\, 0 \leqslant j < p^i \Big\}$$

is a finite set.

The sequences having the finiteness property stated in this theorem are called p-automatic sequences. A full account on this topic and a very complete list of references can be found in the book [3] of Allouche and Shallit.

Concerning algebraic elements in $\mathbb{F}(q)$, a particular subset needs to be considered. An irrational element α in $\mathbb{F}(q)$ is called hyperquadratic, if α^{r+1}, α^r, α, and 1 are linked over $\mathbb{F}_q(T)$, where $r = p^t$, and t is an integer such that $t \geqslant 0$. The subset of all these elements, noted $\mathcal{H}(q)$, contains not only the quadratic ($r = 1$) and the cubic power series ($r = p$ and note that the vector space over $\mathbb{F}_q(T)$ generated by all the α^j's with