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$SO(2n, \mathbb{C})$-character varieties are not varieties of characters

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ABSTRACT

We prove that for $n \geq 2$ and all groups Γ of corank ≥ 2 the coordinate rings of $SO(2n, \mathbb{C})$-character varieties of Γ are not generated by trace functions nor generalized trace functions $\tau_{\phi,\gamma}$, sending $\rho : \Gamma \to SO(2n, \mathbb{C})$ to $tr\phi\rho(\gamma)$, for an arbitrary representation ϕ of $SO(2n, \mathbb{C})$ and any $\gamma \in \Gamma$.
Furthermore, we give examples of non-conjugate completely reducible representations undistinguishable by generalized trace functions. Hence, $SO(2n, \mathbb{C})$-character varieties are not varieties of characters! However, we also prove that any generic $SO(2n, \mathbb{C})$-representation of a free group can be distinguished from all non-equivalent representations by trace functions and by a single generalized trace function.

For a complex, reductive algebraic group G, and a finitely generated discrete group Γ, the G-character variety of Γ, $X_G(\Gamma)$, is the categorical quotient of the representation variety, $Rep(\Gamma, G)$, by the action of G by conjugation, cf. [29,35] as well as for example [14,22,27,31,36] and the references within.[2] (In this paper $Rep(\Gamma, G)$ and $X_G(\Gamma)$ are affine algebraic sets rather than possibly non-reduced schemes of [29,35].) Character

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[2] Throughout the paper, the field of complex numbers can be replaced an arbitrary algebraically closed field of characteristic zero.

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varieties are particularly well studied for $G = SL(2, \mathbb{C})$, due to their applications in low-dimensional topology, cf. for example [2,4–9,11–13,23,24,28,32,34] and for free groups and surface groups Γ, cf. for example [1,10,15,17–21,25,26,30,33,38], often in connection with gauge theory.

In this paper we investigate $SO(2n, \mathbb{C})$-character varieties which were not much studied so far and whose properties, as will be seen below, distinguish them from G-character varieties for other classical groups G.

If G is a matrix group, then every $\gamma \in \Gamma$ defines a trace function

$$\tau_\gamma : X_G(\Gamma) \to \mathbb{C}, \quad \tau_\gamma([\rho]) = tr\rho(\gamma).$$

The G-trace algebra of Γ, denoted by $\mathcal{T}_G(\Gamma)$, is a subalgebra of $\mathbb{C}[X_G(\Gamma)]$ generated by τ_γ, for all $\gamma \in \Gamma$.

For $G = SL(n, \mathbb{C}), Sp(n, \mathbb{C})$ (symplectic groups), and $SO(2n + 1, \mathbb{C})$ (odd special orthogonal groups) the coordinate rings, $\mathbb{C}[X_G(\Gamma)]$, are generated by trace functions, [14,36], and although one might expect that for other groups G as well, it is not the case for $G = SO(2n, \mathbb{C})$. Indeed, if $\rho : \Gamma \to SO(2n, \mathbb{C})$ is irreducible and ρ' is obtained from ρ by a conjugation by a matrix $M \in O(2n, \mathbb{C})$, $detM = -1$, then ρ and ρ' are indistinguishable by trace functions even though it is not difficult to see that they are not $SO(2n, \mathbb{C})$-conjugate and, furthermore, they are distinct in $X_{SO(2n,\mathbb{C})}(\Gamma)$, cf. [37, Sec. 4].

Pursuing better conjugacy invariants of representations, one is naturally lead to the notion of generalized trace functions, which (as an added bonus) are defined irrespectively of specific realizations of G as matrix groups. More specifically, for any $\gamma \in \Gamma$ and any (finite dimensional) representation ϕ of G there is the generalized trace function

$$\tau_{\phi,\gamma} : X_G(\Gamma) \to \mathbb{C}, \quad \tau_{\phi,\gamma}([\rho]) = tr\phi\rho(\gamma).$$

The full G-trace algebra of Γ, $\mathcal{FT}_G(\Gamma)$, is a subalgebra of $\mathbb{C}[X_G(\Gamma)]$ generated by $\tau_{\phi,\gamma}$, for all $\gamma \in \Gamma$ and all ϕ, cf. [36].

We proved in [37] however that even these functions do not generate $\mathbb{C}[X_G(\Gamma)]$ in general. Specifically, for any group Γ of corank ≥ 2 (i.e. having the free group on two generators, F_2, as a quotient) $\mathcal{FT}_{SO(4,\mathbb{C})}(\Gamma)$ is a proper subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(\Gamma)]$. The main result of this paper is a generalization of this statement to even orthogonal groups of higher rank.

Theorem 1.

$$\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma) \subsetneq \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \tag{1}$$

for every group Γ of corank ≥ 2 and all $n \geq 2$.