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Uniform almost relative injective modules

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ABSTRACT

The concept of a module M being almost N-injective where N is some module, was introduced by Baba (1989). For a given module M, the class of modules N, for which M is almost N-injective, is not closed under direct sums. Baba gave a necessary and sufficient condition under which a uniform finite length module U is almost V-injective, where V is a finite direct sum of uniform, finite length modules, in terms of extending properties of simple submodules of V. Let U be a uniform module and V be a finite direct sum of indecomposable modules. Recently, Singh (2016), has determined some conditions under which U is almost V injective, which generalize Baba's result. In the present paper some more results in this direction are proved. A module M is said to be *completely almost self-injective*, if for any two subfactors A, B of M, A is almost B-injective. A necessary and sufficient condition for a module M to be completely almost self-injective is given. Using this, it is proved that a Von Neumann ring R is completely almost right self-injective if and only if $\frac{R}{soc(R_R)}$ is semi-simple and every minimal right ideal of R is injective.

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Introduction

Let M_R, N_R be two modules. As defined by Baba [2], M is said to be *almost N-injective*, if for any homomorphism $f : A \to M$, $A \leqslant N$, either f extends to a homomorphism $g : N \to M$ or there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and a homomorphism $h : M \to N_1$ such that $hf(x) = \pi(x)$ for any $x \in A$, where $\pi : N \to N_1$ is a projection with kernel N_2. This concept plays a significant role is studying extending modules. A module M that is almost M-injective, is called an *almost self-injective module*. For a module M, the class of those modules N for which M is almost N-injective, is not closed under direct sums. Let $\{U_k : 0 \leq k \leq n\}$ be a finite family of uniform modules of finite lengths, and $U = \oplus \sum_{k=1}^{n} U_k$. Baba [2] has given a characterization for U_0 to be almost U-injective, in terms of the property of simple submodules of U being contained in uniform summands of U. Let M be a uniform module and V be a finite direct sum of indecomposable modules. In [7], conditions under which M is almost V-injective have been investigated, thereby Baba's result has been generalized. In Section 1, this study has been further continued. Let M be an almost self-injective, uniform modules. Let T be the set of those maximal homomorphisms $f : N \to M$, $N < M$ which cannot be extended to endomorphisms of M. In Section 1, an algebraic structure on T is given. As defined in Section 2, a module M is called a *completely almost self-injective module*, if for any two subfactors A, B of M, A is almost B-injective. In Theorem 2.4, it is proved that a uniform module U is completely almost self-injective if and only if U is of length not more than 2 and is quasi-injective. In Theorem 2.11 a necessary and sufficient condition for a module M to be completely almost self-injective is proved. From which, it follows that a Von Neumann regular ring R is completely right almost self-injective if and only if $\frac{R}{soc(R_R)}$ is semi-simple and every minimal right ideal of R is injective.

Preliminaries. All rings considered here are with unity and all modules are unital right modules unless otherwise stated. Let M be a module. Then $E(M)$ and $J(M)$ denote the injective hull and radical, respectively, of M. The symbols $N \leqslant M$, $N < M$, $N \subset_e M$ denote that N is a submodule of M, N is a submodule different from M, N is an essential submodule of M respectively. Any submodule of a homomorphic image of M is called a *subfactor* of M. A module M whose ring of endomorphisms $End(M)$ is local, is called an *LE module*. A module M such that its complement submodules are summands of M, is called a *CS module* (or a module satisfying condition (C_1)). The terminology used here is available in standard text books like [1,4]. For details on Von Neumann regular rings see [5].

1. Direct sums of uniform modules

Definition 1.1. Let M_R and N_R be any two modules. Then M is said to be *almost N-injective*, if given any R-homomorphism $f : A \to M$, $A \leqslant N$ either f extends to an R-homomorphism from N to M or there exist a decomposition $N = N_1 \oplus N_2$ with