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Vanishing of relative homology and depth of tensor products

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A B S T R A C T

For finitely generated modules M and N over a Gorenstein local ring R, one has depth M + depth N = depth$(M \otimes_R N)$ + depth R, i.e., the depth formula holds, if M and N are Tor-independent and Tate homology $\widehat{\operatorname{Tor}}_i(M, N)$ vanishes for all $i \in \mathbb{Z}$. We establish the same conclusion under weaker hypotheses: if M and N are $\mathcal{G}$-relative Tor-independent, then the vanishing of $\widehat{\operatorname{Tor}}_i(M, N)$ for all $i \leqslant 0$ is enough for the depth formula to hold. We also analyze the depth of tensor products of modules and obtain a necessary condition for the depth formula in terms of $\mathcal{G}$-relative homology.

1. Introduction

In this paper, R is a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and R-modules are tacitly assumed to be finitely generated.

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In 1961 Auslander [2, 1.2] proved a natural extension of the classical Auslander–Buchsbaum formula. He showed that, if M has finite projective dimension (e.g., R is regular) and M and N are Tor-independent modules, i.e., $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geqslant 1$, then the following remarkable equality holds:

$$\operatorname{depth}_R N = \operatorname{depth}_R(M \otimes_R N) + \operatorname{pd}_R M. \tag{1}$$

Notice, for the special case where $N = R$, equality (1) recovers the Auslander–Buchsbaum formula, i.e., $\operatorname{depth}_R R = \operatorname{depth}_R M + \operatorname{pd}_R M$.

In 1994 Huneke and Wiegand established an important consequence of Tor-independence. They showed equality (1) holds for Tor-independent modules M and N over complete intersection rings R (even if M does not necessarily have finite projective dimension) provided that $\operatorname{pd}_R M$ is replaced by $\operatorname{depth}_R R - \operatorname{depth}_R M$. More precisely, Huneke and Wiegand [16, 2.5] proved that, if R is a complete intersection and M and N are Tor-independent modules, then one has:

$$\operatorname{depth}_R M + \operatorname{depth}_R N = \operatorname{depth}_R R + \operatorname{depth}_R(M \otimes_R N). \tag{2}$$

The equality in (2) is dubbed *the depth formula* by Huneke and Wiegand in [15].

The depth formula is an important tool to study depth of tensor products of modules as well as that of complexes [18]. For example if M and N are maximal Cohen–Macaulay modules (i.e., $\operatorname{depth}_R M = \operatorname{depth}_R N = \dim R$) and the depth formula holds, then R must be Cohen–Macaulay and $M \otimes_R N$ is maximal Cohen–Macaulay. In general it is an open question, even over Gorenstein rings, whether or not the depth formula holds for all Tor-independent modules.

There are quite a few extensions of the aforementioned result of Huneke and Wiegand in the literature; see for example [13]. A substantial generalization in this direction is due to Christensen and Jorgensen: if M has finite $\mathcal{G}$-dimension in the sense of Auslander and Bridger [3] (e.g., R is Gorenstein), then the vanishing of Tate homology $\widehat{\operatorname{Tor}}_i^R(M,N)$ for all $i \in \mathbb{Z}$ (e.g., M has finite projective dimension) is a sufficient condition for the *derived depth formula* to hold; see [12, 2.3]. Noting that the derived depth formula coincides with the depth formula for Tor-independent modules M and N, we observe:

1.1. Theorem. *(Christensen and Jorgensen; see [12, 5.3]) Let M and N be R-modules such that M has finite $\mathcal{G}$-dimension. Then the depth formula holds, i.e.,* $\operatorname{depth}_R M + \operatorname{depth}_R N = \operatorname{depth}_R(M \otimes_R N) + \operatorname{depth} R$, *provided the following conditions hold.*

(i) $\widehat{\operatorname{Tor}}_i^R(M,N) = 0$ *for all* $i \in \mathbb{Z}$.
(ii) $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \geqslant 1$.

The main aim of this article is to obtain a new condition that is sufficient for the depth formula to hold. A new tool we use is the *$\mathcal{G}$-relative homology* $\mathcal{G}\operatorname{Tor}_*^R(M,N)$ which has been defined and studied by Avramov and Martsinkovsky [7], and Iacob [17]; see 2.3 for