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On the deviation and the type of certain local Cohen–Macaulay rings and numerical semigroups

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A B S T R A C T

In J. Herzog and E. Kunz (1973) [6] it was shown that for any pair $(d, t) \in \mathbb{N} \times \mathbb{N}_+$ with $(d, t) \neq (1, 1)$ there exists a local Cohen–Macaulay ring R having deviation $d(R) = d$ and type $t(R) = t$. By E. Kunz (1974) [7] the case $d(R) = 1, t(R) = 1$ cannot occur. In this paper certain Cohen–Macaulay rings are studied for which there are close relations between deviation, type and embedding dimension. Similar relations for other classes of local rings have been proved in the recent paper by L. Sharifan (2014) [15]. Our relations will be applied to numerical semigroups (or equivalently monomial curves) and lead also to some further cases, generalizing E. Kunz (2016) [8] with ring-theoretic proofs, in which a question of H. Wilf (1978) [16] has a positive answer.

1. Introduction

Let R be a Noetherian local ring. Write $\hat{R} = P/I$ where $\hat{R}$ is the completion of R and P is a regular local ring. If $\mu(I)$ denotes the minimal number of generators of I, then the number

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$$d(R) := \mu(I) - (\dim(P) - \dim(R))$$

is independent of the chosen presentation of $\hat{R}$ and is called the *deviation* of R. If $\{z_1, \dots, z_b\}$ is a regular sequence in the maximal ideal $\mathfrak{m}_R$ of R, then $d(R) = d(R/(z_1, \dots, z_b))$ ([5], 1.16 and 3.10b)).

If R is a Cohen–Macaulay ring and $\{z_1, \dots, z_b\}$ a maximal regular sequence in $\mathfrak{m}_R$, hence $b = \dim(R)$, then let $\mathrm{Soc}(R/(z_1, \dots, z_b))$ be the socle of $R/(z_1, \dots, z_b)$, i.e. the $R/\mathfrak{m}_R$-vector space of the elements in $R/(z_1, \dots, z_b)$ annihilated by the maximal ideal $\mathfrak{m}_R$. Then the number

$$t(R) := \dim_{R/\mathfrak{m}_R} \mathrm{Soc}(R/(z_1, \dots, z_b))$$

is independent of the maximal regular sequence and is called the *type* of R.

Informations about the deviation and the type can be found, for example in [6]. In the following special rings R will be studied in which close relations between $d(R), t(R)$ and the embedding dimension $\mathrm{edim}(R)$ of R exist.

2. Relations between deviation and type

Let R be a local Cohen–Macaulay ring of dimension b and embedding dimension $a + b + 1, a \neq 0$. Choose a minimal system $\{x_1, \dots, x_a, y, z_1, \dots, z_b\}$ of generators of $\mathfrak{m}_R$ such that $\{z_1, \dots, z_b\}$ is a regular sequence.

2.1 Theorem. *a) If $x_i x_j \in (y, z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$, then*

$$d(R) \geq \binom{edim(R) - dim(R)}{2} - (edim(R) - dim(R))$$

and

$$t(R) \leq edim(R) - dim(R).$$

b) If even $x_i x_j \in (z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$, then there exists $\delta_1 \in \{0, \dots, a\}$ such that

$$d(R) = \binom{edim(R) - dim(R)}{2} - \delta_1$$

and $\delta_2 \in \{0, 1\}$ such that

$$t(R) = edim(R) - dim(R) - \delta_2.$$

Moreover $\delta_2 = 0$ if and only if $\delta_1 = 0$.
c) If $x_i x_j \in (z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$ and if there exists an element $\sigma \in \mathfrak{m}_R \setminus (x_1, \dots, x_a, z_1, \dots, z_b)$ with $\sigma \cdot (y, x_1, \dots, x_a) \subset (z_1, \dots, z_b)$, then $\delta_1 = \delta_2 = 0$.