Journal of Algebra 478 (2017) 410–418

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Regular subgroups of the affine group with no translations

M.A. Pellegrini *, M.C. Tamburini Bellani

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via Musei 41, I-25121 Brescia, Italy

ARTICLE INFO

Article history:
Received 17 August 2016
Available online xxxx
Communicated by Gernot Stroth

MSC:
20B35
15A63
15A21

Keywords:
Regular subgroup
Affine group
Translations

ABSTRACT

Given a regular subgroup R of $\mathrm{AGL}_n(\mathbb{F})$, one can ask if R contains nontrivial translations. A negative answer to this question was given by Liebeck, Praeger and Saxl for $\mathrm{AGL}_2(p)$ (p a prime), $\mathrm{AGL}_3(p)$ (p odd) and for $\mathrm{AGL}_4(2)$. A positive answer was given by Hegedűs for $\mathrm{AGL}_n(p)$ when $n \geq 4$ if p is odd and for $n = 3$ or $n \geq 5$ if $p = 2$. A first generalization to finite fields of Hegedűs' construction was recently obtained by Catino, Colazzo and Stefanelli. In this paper we give examples of such subgroups in $\mathrm{AGL}_n(\mathbb{F})$ for any $n \geq 5$ and any field $\mathbb{F}$. For $n < 5$ we provide necessary and sufficient conditions for their existence, assuming R to be unipotent if $\operatorname{char} \mathbb{F} = 0$.

1. Introduction

Consider the affine group

$$\mathrm{AGL}_n(\mathbb{F}) = \left\{ \begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} : v \in \mathbb{F}^n,\ A \in \mathrm{GL}_n(\mathbb{F}) \right\}$$

* Corresponding author.
E-mail addresses: marcoantonio.pellegrini@unicatt.it (M.A. Pellegrini), mariaclara.tamburini@gmail.com (M.C. Tamburini Bellani).

http://dx.doi.org/10.1016/j.jalgebra.2017.01.045
0021-8693/

acting on the right on the row vector space $\mathbb{F}^{n+1}$, whose canonical basis will be denoted by $\{e_0, e_1, \dots, e_n\}$. Furthermore, denote by

$$\pi : \mathrm{AGL}_n(\mathbb{F}) \to \mathrm{GL}_n(\mathbb{F})$$

the obvious epimorphism $\begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} \mapsto A$, whose kernel is the translation group $\mathcal{T}$. A subgroup R of $\mathrm{AGL}_n(\mathbb{F})$ is called *regular* if it acts regularly on the set of the affine points: namely if, for every $v \in \mathbb{F}^n$, there exists a unique element of R having the affine point $(1, v)$ as first row.

The problem of the existence of regular subgroups of $\mathrm{AGL}_n(\mathbb{F})$ having no translations other than the identity was first raised by Liebeck, Praeger and Saxl in [4]. Clearly, for such subgroups we have

$$R \cong \pi(R). \tag{1.1}$$

In the case of fields $\mathbb{F} = \mathbb{F}_p$ of prime order p, the above-mentioned authors also proved that no such regular subgroups exist for $\mathrm{AGL}_2(p)$, any p, $\mathrm{AGL}_3(p)$, $p > 2$, and for $\mathrm{AGL}_4(2)$. The first positive examples, which proved their existence, were constructed by Hegedűs in [2], in the case $\mathbb{F} = \mathbb{F}_p$. More precisely, he proved that $\mathrm{AGL}_n(p)$ contains a regular subgroup having no translations other than the identity, whenever (i) $n = 3$ or $n \geq 5$, if $p = 2$ or (ii) $n \geq 4$, if $p > 2$. The crucial property that he used is the existence of a non-degenerate quadratic form over $\mathbb{F}_p^{n-1}$ and an embedding of the additive group $(\mathbb{F}_p, +)$ into the corresponding orthogonal group. Clearly this property holds for much more general fields than $\mathbb{F}_p$. This fact was recently used by Catino, Colazzo and Stefanelli who extended Hegedűs' result to $\mathbb{F} = \mathbb{F}_{p^\ell}$, [1].

In Section 2 we extend to an arbitrary field $\mathbb{F}$ the negative results of [4], giving an independent proof of the following facts:

Theorem 1.1. *Let R be a regular subgroup of $\mathrm{AGL}_n(\mathbb{F})$ and suppose that R is unipotent if $\mathrm{char}\,\mathbb{F} = 0$. Assume that one of the following conditions holds:*

(i) $n \leq 2$;
(ii) $n = 3$ *and* $\mathbb{F} \neq \mathbb{F}_2$;
(iii) $n = 4$ *and* $\mathrm{char}\,\mathbb{F} = 2$.

Then R contains a nontrivial translation.

On the other hand, using Hegedűs' method, we generalize [1] proving the following result.

Theorem 1.2. *Let $\mathbb{F}$ be any field and W be a subspace of $\mathbb{F}$, viewed as a vector space over its prime field $\mathbb{F}_0$. Assume that one of the following conditions holds:*