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Rings in which every ideal is pure-projective or FP-projective ☆

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ABSTRACT

A ring R is said to be left pure-hereditary (resp. RD-hereditary) if every left ideal of R is pure-projective (resp. RD-projective). In this paper, some properties and examples of these rings, which are nontrivial generalizations of hereditary rings, are given. For instance, we show that if R is a left RD-hereditary left nonsingular ring, then R is left Noetherian if and only if $\text{u.dim}({}_RR) < \infty$. Also, we show that a ring R is quasi-Frobenius if and only if R is a left FGF, left coherent right pure-injective ring. A ring R is said to be left FP-hereditary if every left ideal of R is FP-projective. It is shown that if R is a left CF ring, then R is left Noetherian if and only if R is left pure-hereditary, if and only if R is left FP-hereditary, if and only if R is left coherent. It is shown that every left self-injective left FP-hereditary ring is semiperfect. Finally, it is shown that a ring R is left FP-hereditary (resp. left coherent) if and only if every submodule (resp. finitely generated submodule) of a projective left R-module is FP-projective, if and only if every pure factor module of an injective left R-module is injective

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(resp. FP-injective), if and only if for each FP-injective left R-module U, $E(U)/U$ is injective (resp. FP-injective).

1. Introduction

According to Warfield's criterion [22, Corollary 3], pure-projective modules can be defined as direct summands of direct sums of finitely presented modules. A left R-module M is called *FP-injective* (or absolutely pure) [11] if it is pure in every left R-module that contains it. A left R-module P is called *FP-projective* [13] if $\mathrm{Ext}^1_R(P, M) = 0$, for each FP-injective left R-module M.

A ring R is called *left hereditary* if every left ideal of R is projective. Since pure-projective modules and FP-projective modules are generalizations of projective modules, it seems natural to ask "When every left ideal of R is pure-projective or FP-projective". We refer to rings with this property as *left pure-hereditary rings* and *left FP-hereditary rings*, respectively. Examples of left pure-hereditary rings and left FP-hereditary rings are left Noetherian rings and left hereditary rings (see Remark 2.3). In this paper, we shall study left pure-hereditary rings and left FP-hereditary rings which are nontrivial generalizations of left hereditary rings.

Obviously, every pure-projective module is FP-projective. But the converse is not true, in general. Consequently, we have the following relationships:

$$\{\text{hereditary rings}\} \subset \{\text{pure-hereditary rings}\} \subset \{\text{FP-hereditary rings}\}.$$

In Section 2, we investigate the rings in which every ideal is pure-projective or RD-projective (i.e., a direct summand of a direct sum of cyclically presented modules [22]). A ring R is called *left pure-hereditary* (resp. *left RD-hereditary*) if every left ideal of R is pure-projective (resp. RD-projective). First, we show that the notions of pure-hereditary rings and RD-hereditary rings are nontrivial generalizations of hereditary rings, pure-semisimple rings, Noetherian rings and principal ideal rings (Remark 2.3 and Example 2.4). Also, we investigate the rings over which the class of left pure-hereditary (or RD-hereditary) rings coincides with the class of left hereditary rings (Propositions 2.5, 2.6 and 2.9, Corollaries 2.7 and 2.16 and Theorem 2.17). F.L. Sandomierski proved that if R is a left hereditary ring, then R is left Noetherian if and only if $\mathrm{u.dim}({}_RR) < \infty$. In Theorem 2.10, we generalize Sandomierski's theorem. In fact, we show that if R is a left RD-hereditary left nonsingular ring, then R is left Noetherian if and only if $\mathrm{u.dim}({}_RR) < \infty$. It is also shown that if R is a left Artinian ring such that every maximal left ideal of R is p-injective, then R is left RD-hereditary if and only if every maximal left ideal of R is RD-projective (Theorem 2.12). It is well known that a ring R is left hereditary if and only if every factor module of an injective left R-module is injective. In Proposition 2.13, we obtain an analogue of this fact. We show that if R is a