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# Ideal theory of infinite directed unions of local quadratic transforms

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Let $(R, \mathfrak{m})$ be a regular local ring of dimension at least 2. Associated to each valuation domain birationally dominating $R$, there exists a unique sequence $\{R_n\}$ of local quadratic transforms of $R$ along this valuation domain. We consider the situation where the sequence $\{R_n\}_{n\geq 0}$ is infinite, and examine ideal-theoretic properties of the integrally closed local domain $S = \bigcup_{n\geq 0} R_n$. Among the set of valuation overrings of $R$, there exists a unique limit point $V$ for the sequence of order valuation rings of the $R_n$. We prove the existence of a unique minimal proper Noetherian overring $T$ of $S$, and establish the decomposition $S = T \cap V$. If $S$ is archimedean, then the complete integral closure $S^*$ of $S$ has the form $S^* = W \cap T$, where $W$ is the rank 1 valuation overring of $V$.

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## 1. Introduction

Let $(R, \mathfrak{m})$ be a $d$-dimensional regular local ring with $d \geq 2$. The morphism $\phi$ : Proj $R[\mathfrak{m}t] \to \operatorname{Spec} R$ defines the blow-up of the maximal ideal $\mathfrak{m}$ of $R$. Let $(R_1, \mathfrak{m}_1)$ be the local ring of a point in the fiber of $\mathfrak{m}$ defined by $\phi$. Then $R_1$, called a *local quadratic transform* of $R$, is a regular local ring of dimension at most $d$ that birationally dominates $R$. Local quadratic transforms have historically played an important role in resolution of singularities and in the understanding of regular local rings. Classically, Zariski's unique factorization theorem for ideals in a 2-dimensional regular local ring [36] relies on local quadratic transforms in a fundamental way. The 2-dimensional regular local rings birationally dominating $R$ are all iterated local quadratic transforms of $R$, and they are in one-to-one correspondence with the simple complete $\mathfrak{m}$-primary ideals of $R$. More recently, Lipman [24] uses similar methods to prove a unique factorization theorem for a special class of complete ideals in regular local rings of dimension $\geq 2$.

By taking regular local rings of dimension at least 2, iteration of the process of local quadratic transforms yields an infinite sequence $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$. We consider the directed union of this infinite sequence of local quadratic transforms, and set $S = \bigcup_{n \geq 0} R_n$. Since the rings $R_n$ are local rings that are linearly ordered under domination, $S$ is local, and since the rings $R_n$ are integrally closed, so is $S$. However, if $S$ is not a discrete valuation ring, then $S$ is not Noetherian. On the other hand, one may consider a valuation ring $(V, \mathfrak{N})$ that dominates $R$. There is a unique local quadratic transform $R_1$ of $R$ that is dominated by $V$, called the *local quadratic transform of $R$ along $V$*. If $V$ is the order valuation ring of $R$, then $R_1 = V$, but otherwise, one may take the local quadratic transform $R_2$ of $R_1$ along $V$. Specifically, $R_1 = R[\mathfrak{m}/x]_{\mathfrak{N} \cap R[\mathfrak{m}/x]}$, where $x \in \mathfrak{m}$ is such that $xV = \mathfrak{m}V$.

Iterating this process yields a possibly infinite sequence $\{(R_n, \mathfrak{m}_n)\}$ of local quadratic transforms, where this process terminates if and only if $V$ is the order valuation ring of some $R_n$. Abhyankar [1, Proposition 4] proves that this sequence is finite if and only if the transcendence degree of $V/\mathfrak{N}$ over $R/\mathfrak{m}$ is $d-1$ (that is, by the dimension formula [25, Theorem 15.5], the residual transcendence degree of $V/\mathfrak{N}$ is as large as possible; in such a case $V$ is said to be a *prime divisor* of $R$). Otherwise, the induced sequence is infinite, and it is in this case that we are especially interested in this article.

Abhyankar [1, Lemma 12] proves that if $\dim R = 2$, then the union $\bigcup_{i \geq 0} R_i$ is equal to $V$. For the setting where $\dim R > 2$, Shannon [35] presents several examples showing that the directed union $S = \bigcup_{i \geq 0} R_i$ is properly contained in $V$, and in particular $S$ is not a valuation ring. More generally, Lipman [24, Lemma 1.21.1] observes that if $P$ is a nonmaximal prime ideal of the regular local ring $R$, then there exists an infinite sequence of local quadratic transforms of $R$ whose union $S$ is contained in $R_P$. Thus if we take $P$ so that $\operatorname{ht} P > 1$, then $S$ cannot be a valuation ring, since the overring $R_P$ of $S$ is not a valuation ring. Thus arises the question of the nature of $S$ when $S$ is not a valuation ring.

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