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Products of test elements of free factors of a free group ☆

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Let G be a group. An element $g \in G$ is called a test element of G if for every endomorphism $\varphi : G \to G$, $\varphi(g) = g$ implies that φ is an automorphism. Let $F(X)$ be a free group on a finite non-empty set X, and let $X = X_1 \coprod X_2 \coprod \ldots \coprod X_r$ be a finite partition of X into $r \geq 2$ non-empty subsets. For $i = 1, 2, \ldots, r$, let $u_i \in \langle X_i \rangle \leq F(X)$, and let $w(z_1, \ldots, z_r)$ be a word in the variables $z_1, \ldots, z_r$. We give several sufficient conditions on u_i $(1 \leq i \leq r)$ and w for $w(u_1, \ldots, u_r)$ to be a test element of $F(X)$. As an application of these results, we give examples of test elements of a free group of rank greater than two that are not test elements in any pro-p completion of the group.

1. Introduction

Let G be a group. An element $g \in G$ is called a test element of G if for every endomorphism $\varphi : G \to G$, $\varphi(g) = g$ implies that φ is an automorphism. The notion of a test element was introduced by Shpilrain [10]. Clearly, this concept could be considered

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in any concrete category. Indeed, test elements have been studied in other categories of algebraic structures, such as polynomial algebras and Lie algebras (see [6] and the references therein).

Throughout the paper, we denote by $F(X)$ or $F(x_1, \dots, x_n)$ a free group with basis $X = \{x_1, \dots, x_n\}$. The first non-trivial example of a test element of a free group was obtained by Nielsen [7], who proved that the commutator $[x_1, x_2]$ is a test element of $F(x_1, x_2)$. Zieschang [15] generalized this result by proving that $[x_1, x_2][x_3, x_4] \dots [x_{2m-1}, x_{2m}]$ is a test element of $F(x_1, x_2, \dots, x_{2m})$. Further, Rips [9] showed that every higher commutator of weight n (with arbitrary disposition of commutator brackets) involving all n letters $x_1, \dots, x_n$ is a test element of $F(x_1, \dots, x_n)$. In another direction, Turner [14] proved that $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ is a test element of $F(x_1, \dots, x_n)$ if and only if $k_i \neq 0$ for all $1 \leq i \leq n$ and $\gcd(k_1, \dots, k_n) \neq 1$ (see also [16]). Another family of test elements of free groups was described by Dold [3]: A k-taut word $w \in F(X)$ ($k \neq 1$, $|X| < \infty$) is a test element of $F(X)$ if the corresponding cyclic graph has one vertex.

Let G be a group. A subgroup R of G is a *retract* of G if there exists a homomorphism $r : G \to R$, called *retraction*, which restricts to the identity on R. An important step in the study of test elements was made by Turner [14], who gave the following characterization of test elements in free groups.

Theorem 1.1 *([14]). An element in a free group F of finite rank is a test element of F if and only if it is not contained in any proper retract of F.*

For each $1 \leq i \leq n$, let $\sigma_{x_i} : F(x_1, \dots, x_n) \to \mathbb{Z}$ be the homomorphism defined by $\sigma_{x_i}(x_j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. Note that for $w \in F(x_1, \dots, x_n)$, $\sigma_{x_i}(w)$ is just the sum of the exponents of all occurrences of x_i in w.

A proper retract of a free group of rank n has rank at most $n - 1$ (see Proposition 4.1 i)). Thus every proper retract of a free group of rank two is cyclic. Moreover, it is easy to see that $\langle w \rangle \leq F(x_1, \dots, x_n)$ is a retract of $F(x_1, \dots, x_n)$ if and only if $\gcd(\sigma_{x_1}(w), \dots, \sigma_{x_n}(w)) = 1$. These observations lead to a complete description of test elements of a free group of rank two.

Theorem 1.2 *([14]). Let $w \in F(x_1, \dots, x_n)$, and $m \in \mathbb{Z} \setminus \{0\}$. Then w^m is a test element of F if and only if w is a test element of F. If $n = 2$ and w is not a proper power, then w is a test element of $F(x_1, x_2)$ if and only if $\gcd(\sigma_{x_1}(w), \sigma_{x_2}(w)) \neq 1$.*

Unfortunately, very little is known about retracts of free groups of rank greater than two, and no description (along the lines of Theorem 1.2) of test elements of free groups of higher rank is known.

Let G be a group and $w \in F(z_1, \dots, z_n)$. Given $g_1, \dots, g_n \in G$, we denote by $w(g_1, \dots, g_n)$ the image of w in G under the homomorphism from $F(z_1, \dots, z_n)$ to G defined by $z_i \to g_i$, $1 \leq i \leq n$. In this paper, we consider the following problem.