Accepted Manuscript

Representations of McLain groups

Fernando Szechtman, Allen Herman, Mohammad A. Izadi

PII: S0021-8693(16)30416-1
DOI: http://dx.doi.org/10.1016/j.jalgebra.2016.10.035
Reference: YJABR 15971

To appear in: *Journal of Algebra*

Received date: 18 June 2015

Please cite this article in press as: F. Szechtman et al., Representations of McLain groups, *J. Algebra* (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.10.035

REPRESENTATIONS OF MCLAIN GROUPS

FERNANDO SZECHTMAN, ALLEN HERMAN, AND MOHAMMAD A. IZADI

Abstract. Basic modules of McLain groups $M = M(\Lambda, \leq, R)$ are defined and investigated. These are (possibly infinite dimensional) analogues of André's supercharacters of $U_n(q)$. The ring R need not be finite or commutative and the field underlying our representations is essentially arbitrary: we deal with all characteristics, prime or zero, on an equal basis. The set Λ, totally ordered by $\leq$, is allowed to be infinite. We show that distinct basic modules are disjoint, determine the dimension of the endomorphism algebra of a basic module, find when a basic module is irreducible, and exhibit a full decomposition of a basic module as direct sum of irreducible submodules, including their multiplicities. Several examples of this decomposition are presented, and a criterion for a basic module to be multiplicity-free is given. In general, not every irreducible module of a McLain group is a constituent of a basic module.

1. Introduction

In 1954 McLain [M] constructed a family of groups that has been a rich source of examples in group theory ever since (see [M2], [R], [HH], [Ro], [W], [DG], [CS], [Sz2], for instance). A general McLain group $M = M(\Lambda, \leq, R)$ depends on a set Λ, partially ordered by $\leq$, and an arbitrary ring R with $1 \neq 0$. Even though a partial order will do for some of our purposes, for best results a total order will be required. In the special case when $|\Lambda| = n$ is finite and $\leq$ is a total order, $M = U_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ of all upper triangular matrices with 1's on the main diagonal.

The main goal of this paper is define and study basic modules of M, which are a generalization of the supercharacters of $U_n(q)$, where $R = F_q$ is a finite field of characteristic p. We stress the fact that Λ as well as R are allowed to be infinite, and M-modules are allowed to be infinite dimensional over an arbitrary field F (which need not have characteristic 0). Moreover, the commutativity or not of R plays no role whatsoever, so we will allow R to be non-commutative. It is perhaps surprising how of much of the theory of supercharacters goes through in this context. A detailed description appears below, after an overview of prior work on the subject.

The representation theory of $U_n(q)$ draws considerable attention due to its attractive nature and open problems. The literature on the subject, as well as on the related algebra groups and Sylow p-subgroups of classical groups, is too vast to review in full detail and we will restrict ourselves to a limited overview.

One line of investigation was concerned with the degrees of the complex irreducible characters of $U_n(q)$. In 1974 Lehrer [L] considered the so called elementary characters of $U_n(q)$ as well as certain products of them, obtaining ([L, Corollary

2010 *Mathematics Subject Classification.* 20C15, 20C20.

Key words and phrases. McLain group; unitriangular group; basic character; supercharacter.

The first and second authors were supported in part by an NSERC discovery grant.