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Refinements of Dade's Projective Conjecture for p-solvable groups

Alexandre Turull

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

Dade's Projective Conjecture is known to be true for finite p-solvable groups thanks to work of G.R. Robinson, but remains open in general. Work of Isaacs and Navarro suggested to Uno and Boltje refinements of this conjecture. These refinements were studied for finite p-solvable groups by Glesser. In the present paper, inspired by earlier work of Turull, we propose further refinements of the conjecture that take into account the Schur indices and the elements of the Brauer group. We prove that all these refinements of Dade's Projective Conjecture hold for all finite p-solvable groups. In particular, we obtain that the version of Dade's Projective Conjecture which involves character degree residues modulo p, fields of definition and Schur indices, as well as the full strength of Boltje's Conjecture both hold for all finite p-solvable groups. The proof develops a Clifford theory for normalizers of chains of p-subgroups which allows one to reduce the calculation of the relevant sums to simpler groups.

E-mail address: turull@ufl.edu.

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1. Introduction

Dade's Projective Conjecture [2] is closely connected to the McKay Conjecture and to Alperin's Weight Conjecture, and it has been the object of intense study. While it is known to be true for all p-solvable groups [10], it remains open in general.

There is a feeling that these conjectures reflect some more fundamental and precise properties of the representations of finite groups. Furthermore, there is the hope that more precise formulations of these properties could lead to their own proof and, consequently, the proof of the conjectures.

Strengthenings of the McKay Conjecture and the Alperin Weight Conjecture were proposed by Isaacs and Navarro [6], by Navarro [9], and by Turull [12]. The McKay Conjecture with all these strengthenings was proved to be true for all p-solvable groups by Turull [15]. The Alperin Weight Conjecture with the Navarro strengthening was proved to be true for all p-solvable groups by Turull [16]. In fact, both these papers prove stronger statements than simply the strengthened McKay or Alperin Weight Conjecture for p-solvable groups because they also prove some further conditions which, although they hold for all finite p-solvable groups, are known not to be true for arbitrary finite groups. The Isaacs and Navarro strengthening of the McKay Conjecture was adapted to the point of view of Dade's Conjectures by Uno [19,20]. Boltje [1] proposed a new conjecture which would imply strengthened versions of Dade's Projective Conjecture and also perhaps suggest a cohomological reason as to why it may be true. Surprisingly, the Turull strengthening has not been adapted to the point of view of Dade's conjectures until the present paper.

These conjectures are known to be true in many special cases. Many of them have been proved to be true for certain simple groups, or groups close to simple groups. And efforts are under way to extract a proof of some of these conjectures by ultimately reducing them to questions about finite simple groups.

For p-solvable groups, as mentioned above, we know that Dade's Projective Conjecture holds because of work of Robinson [10]. Robinson's approach is based on work of Külshammer and Puig [7] on extensions of nilpotent blocks. In this approach, the characters in a block that lie above a fixed character $\lambda \in \mathrm{Irr}(\mathrm{Z}(G) \cap \mathrm{O}_p(G))$ are not treated individually. However, this obstacle is overcome by noticing that the relevant alternating sums required in the Dade Projective Conjecture for any two Galois conjugate λ's will be the same, and using induction. Hence, the full proof of Dade's Projective Conjecture for all p-solvable groups is obtained indirectly.

Glesser [5] studied the Boltje refinements of Dade's Projective Conjecture for p-solvable groups, and obtained for these groups refinements of the original Dade Projective Conjecture. The proof is based on Robinson's approach [10]. In particular, it ultimately relies on Külshammer and Puig [7], and on an indirect treatment of the characters in a block that lie above a fixed character $\lambda \in \mathrm{Irr}(\mathrm{Z}(G) \cap \mathrm{O}_p(G))$. Glesser is not able to count characters lying over a given $\lambda \in \mathrm{Irr}(\mathrm{Z}(G) \cap \mathrm{O}_p(G))$. Glesser obtains the version of Dade's Projective Conjecture which involves character degree residues