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Permutability of injectors with a central socle in a finite solvable group

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ABSTRACT

In response to an Open Question of Doerk and Hawkes [5, IX Section 3, page 615], we shall show that if $\mathcal{Z}^\pi$ is the Fitting class formed by the finite solvable groups whose π-socle is central (where π is a set of prime numbers), then the $\mathcal{Z}^\pi$-injectors of a finite solvable group G permute with the members of a Sylow basis in G. The proof depends on the properties of certain extraspecial groups [4].

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1. Introduction

Throughout this Introduction, let H be a subgroup of a finite solvable group G, and let $\pi = \{p_1, p_2, \ldots, p_m\}$ be a set of prime numbers. We use the notation of Doerk and Hawkes [5], and as in [3], we define $\mathcal{Z}^\pi$ to be the class of finite solvable groups H such that $\operatorname{Soc}_\pi H \leq \mathbf{Z}(H)$, and as before write $\mathcal{Z}^p = \mathcal{Z}^{\{p\}}$. These classes are Fitting classes, so that any finite solvable group possesses a conjugacy class of injectors for any given such class. In [3], we described inductive methods for constructing $\mathcal{Z}^\pi$-injectors. In this work, we prove that these injectors are permutable. This means that for any such injector H there exists a *Sylow basis* Σ in G such that H permutes with every element of Σ, where a Sylow basis is a set of Sylow subgroups of G, with $|\Sigma \cap \operatorname{Syl}_p G| = 1$ for each prime number p, such that all pairs of members of Σ permute with each other [5, I(4.7)]. Doerk and Hawkes characterize the property of permutability as the one that separates manageable from unmanageable Fitting classes [5, p. 615], making the often difficult determination of permutability of a Fitting class the key to obtaining a thorough analysis of its properties. We prove:

Theorem. *If π is a set of prime numbers, and G is a finite solvable group, then the $\mathcal{Z}^\pi$-injectors of G are system permutable in G.*

Corollary. *Let $G = HK$ be a solvable semidirect product, with $K \lhd G$ and $H \cap K = 1$, and suppose U is an $\mathbf{F}_pG$-module (where p is a prime number, and $\mathbf{F}_p$ is the field of order p). Choose a Sylow p-subgroup P of H, and assume that $p \nmid |K|$. Let $\operatorname{Soc}_{\mathbf{F}_pG} U$ be the socle of U (generated by the minimal submodules). If $\mathbf{C}_G(\operatorname{Soc}_{\mathbf{F}_pG} U) = 1$ and $\mathbf{C}_G(\mathbf{C}_U(P)) = H$, then there is a Sylow basis Σ of K, such that H normalizes each subgroup in Σ.*

Proof. We can deduce the Corollary from the Theorem, using some of the results quoted in Section 2, as follows. Form the natural semidirect products

$$G_0 = GU, \quad H_0 = HU, \quad K_0 = KU, \quad P_0 = PU.$$

Then $U = \mathbf{C}_{G_0}(\operatorname{Soc}_{\mathbf{F}_pG_0} U)$, so U is the $\mathcal{Z}^p$-radical of G_0 by Lemma 2.7(c). Also P_0 is a Sylow p-subgroup of G_0 and $H_0 = \mathbf{C}_{G_0}(\mathbf{C}_U(P_0))$, so H_0 is a $\mathcal{Z}^p$-injector of G_0 by Lemma 2.7(f) (or [5, IX(4.19)]). Hence H_0 is system permutable in G_0 by the Theorem, and it follows from Lemma 2.4(d) that $H \cong H_0/U$ normalizes a Sylow basis Σ in $K \cong K_0/U$. $\square$

The lay-out of the paper is as follows: In Section 2 we state some known results and prove results on a variety of topics for later use, and in Section 3 we quote some results about extraspecial groups [4]. We begin a general study of counterexamples to permutability claims for injectors in Section 4, introducing the specific case of permutability of $\mathcal{Z}^\pi$-injectors in Section 5.

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