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Algebraic subgroups of acylindrically hyperbolic groups

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A subgroup of a group G is called *algebraic* if it can be expressed as a finite union of solution sets to systems of equations. We prove that a non-elementary subgroup H of an acylindrically hyperbolic group G is algebraic if and only if there exists a finite subgroup K of G such that $C_G(K) \leq H \leq N_G(K)$. We provide some applications of this result to free products, torsion-free relatively hyperbolic groups, and ascending chains of algebraic subgroups in acylindrically hyperbolic groups.

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1. Introduction

Let G be a group and $\langle x \rangle$ be the infinite cyclic group generated by x. For each $g \in G$, let φ_g denote the homomorphism $G * \langle x \rangle \to G$ induced by taking the identity map on G and sending $x \mapsto g$. Given an element $w(x)$ of the free product $G * \langle x \rangle$, there is a corresponding function $G \to G$ whose evaluation at $g \in G$ is

$$w(g) := \varphi_g(w(x)).$$

Building on this notion, we write $w(x) = 1$ to represent a *group equation* in the single variable x with coefficients in G whose *solution set* is

$$\{g \in G \mid w(g) = 1\}.$$

The solution set above is called the *primitive solution set corresponding to* $w(x)$.

Definition 1.1. The *Zariski topology* (or *verbal topology* as in [4]) on G is defined by taking the collection of primitive solution sets to be a sub-basis for the closed sets of the topology. That is, each Zariski-closed set of G is of the form

$$\cap_{i \in I} S_i$$

for some indexing set I, where for each $i \in I$, the set S_i is a finite union of primitive solution sets corresponding to (single-variable) group equations with coefficients in G. Note that in general, the Zariski topology is not a group topology.

The Zariski topology (and its more general form in [1]) is useful in bringing notions from algebraic geometry to the group theoretical setting in order to aid in the study of equations over groups. The topology is also useful in the study of topological groups: Zariski-closed sets are closed in every T_0 group topology, and, in the case of countable groups, the Zariski-closed sets are the only such sets [12]. For some recent applications to topological groups, see [11] and references therein.

The following definition is motivated by the standard notion of an algebraic subgroup in algebraic geometry.

Definition 1.2. A Zariski-closed subgroup (or more generally, a subset) of G is called *algebraic*.