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Multiplicity and invariants in birational geometry

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ABSTRACT

The multiplicity of a point on a variety is a fundamental invariant to estimate how the singularity is bad. It is introduced in a purely algebraic context. On the other hand, we can also attach to the singularity the log canonical threshold and the minimal log discrepancy, which are introduced in a birational theoretic context. In this paper, we show bounds of the multiplicity by functions of these birational invariants for a singularity of locally a complete intersection. As an application, we obtain the affirmative answer to Watanabe's conjecture on the multiplicity of canonical singularity of locally a complete intersection up to dimension 32.

1. Introduction

The multiplicity of a point on a variety is a fundamental invariant to estimate how the singularity is bad. Among many results on multiplicities, Kei-ichi Watanabe proved the following bound of multiplicity for the invariant ring by a finite group action [8]:

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Theorem 1.1. *If $G \subset \mathrm{SL}(n, \mathbb{C})$ is a finite abelian group and if the invariant ring $\mathbb{C}[[x_1, \ldots, x_n]]^G$ by the action of G is a complete intersection, then* $\mathrm{mult}\mathbb{C}[[x_1, \ldots, x_n]]^G \leq 2^{n-1}$.

This shows that the multiplicity of an n-dimensional quotient singularity of locally a complete intersection is less than or equal to 2^{n-1}. Note that if a variety is locally a complete intersection with quotient singularities, it has canonical singularities. The following conjecture was posed by Watanabe as a generalization of Theorem 1.1.

Conjecture 1.2. *Let X be an n-dimensional variety of locally a complete intersection with canonical singularities. Then*

$$\mathrm{mult}_x X \leq 2^{n-1}$$

for a closed point x of X.

When $n = 2$, the conjecture holds true. Actually, two dimensional canonical singularities are just A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$ type singularities, i.e., rational double singularities. However, as there is no classification of higher dimensional canonical singularities of locally a complete intersection, we have to make a different approach to this problem. First observation in this paper is about a hypersurface singularities.

Proposition 1.3. *Let X be a hypersurface and let $x \in X$ be a closed point. Then* $\mathrm{mult}_x X \leq n + 1 - \mathrm{mld}_x(X)$.

This proposition shows that the conjecture is true for a hypersurface with canonical singularities, since $n + 1 - \mathrm{mld}_x(X) \leq 2^{n - \mathrm{mld}_x(X)}$ and $\mathrm{mld}_x(X) \geq 1$. Moreover this proposition implies that there exists some relation between the multiplicity and the minimal log discrepancy of a hypersurface at a closed point. This observation and many other examples suggest us the following prediction:

Conjecture 1.4.

(1) *Let X be an n-dimensional variety of locally a complete intersection. Then*

$$\mathrm{mult}_x X \leq 2^{n - \mathrm{mld}_x(X)}$$

for a closed point x of X and the equality holds if and only if $\mathrm{emb}(X, x) = 2n - \mathrm{mld}_x(X)$.

(2) *Let X be an n-dimensional variety of locally a complete intersection with log canonical singularities. Then*

$$\mathrm{mult}_x X \leq 2^{n - \lceil \mathrm{lct}(\mathfrak{m}_x) \rceil}$$