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Generalized Demazure modules and fusion products

B. Ravinder

Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India

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ABSTRACT

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with highest root θ and let $\mathfrak{g}[t]$ be the corresponding current algebra. In this paper, we consider the $\mathfrak{g}[t]$-stable Demazure modules associated to integrable highest weight representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. We prove that the fusion product of Demazure modules of a given level with a single Demazure module of a different level and with highest weight a multiple of θ is a generalized Demazure module, and also give defining relations. This also shows that the fusion product of such Demazure modules is independent of the chosen parameters. As a consequence we obtain generators and relations for certain types of generalized Demazure modules. We also establish a connection with the modules defined by Chari and Venkatesh.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $\widehat{\mathfrak{g}}$ the corresponding affine Lie algebra. In this paper, we are interested in Demazure modules associated to integrable highest weight representations of $\widehat{\mathfrak{g}}$. These modules, which are actually modules for a Borel subalgebra of $\widehat{\mathfrak{g}}$, are indexed by a dominant integral affine weight

E-mail address: ravinder@math.tifr.res.in.

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and an element of the affine Weyl group. We are mainly interested in the Demazure modules which are preserved by a maximal parabolic subalgebra containing the Borel. The maximal parabolic subalgebra of our interest is the current algebra $\mathfrak{g}[t]$, which is the algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$ with the obvious point-wise bracket. Equivalently, it is the complex vector space $\mathfrak{g} \otimes \mathbb{C}[t]$ with Lie bracket being the $\mathbb{C}[t]$-bilinear extension of the Lie bracket on $\mathfrak{g}$. The degree grading on $\mathbb{C}[t]$ gives a natural $\mathbb{Z}$-grading on $\mathfrak{g}[t]$ and makes it a graded Lie algebra. The $\mathfrak{g}[t]$-stable Demazure modules are known to be indexed by pairs (ℓ, λ), where ℓ is the level of the integrable representation of $\widehat{\mathfrak{g}}$ and λ is a dominant integral weight of $\mathfrak{g}$. We denote the corresponding module by $D(\ell, \lambda)$. These are in fact finite-dimensional graded $\mathfrak{g}[t]$-modules.

A powerful tool to study the category of finite-dimensional graded $\mathfrak{g}[t]$-modules is the fusion product, which was introduced by Feigin and Loktev in [5]. Although the fusion product is by definition dependent on a choice of parameters, it is widely expected that it will turn out to be independent of the choices, and in several cases this has been proved (see [3,4,7,10,13,14,16]). It is proved in [3] that the fusion product of Demazure modules of a given level is again a Demazure module of the same level. In [14], for $\mathfrak{g}$ simply laced, it is proved that the fusion product of Demazure modules of different level with highest weight a multiple of a fundamental weight is a generalized Demazure module, and used this to solve the $X = M$ conjecture. The generalized Demazure modules are indexed by p dominant integral affine weights and p affine Weyl group elements, where $p \geq 1$. Their defining relations are not known except when $p = 1$, where they are actually the Demazure modules. But in special cases the character of generalized Demazure modules is known in terms of the Demazure operators [11,15].

In this paper, we investigate further the fusion product of different level Demazure modules. Let θ be the highest root of $\mathfrak{g}$. We consider the fusion product of Demazure modules of a given level with a single Demazure module of a different level and with highest weight a multiple of θ. We prove that this fusion product as a $\mathfrak{g}[t]$-module is isomorphic to a generalized Demazure module, and give the defining relations. More precisely, given positive integers k, ℓ, m such that $\ell \geq m \geq k$, and a sequence of dominant integral weights $\lambda_1, \ldots, \lambda_p$ of $\mathfrak{g}$, we prove that the fusion product

$$D(\ell, \ell\lambda_1) * \cdots * D(\ell, \ell\lambda_p) * D(m, k\theta), \tag{1.1}$$

of Demazure modules is a generalized Demazure module, and also give the defining relations (see Theorem 3.3). This also proves that the fusion product (1.1) is independent of the chosen parameters.

Our main results (Theorems 3.2 and 3.3) enable us to obtain short exact sequences of fusion products and generalized Demazure modules (see Corollary 3.4), and a surjective morphism between two fusion products (see Corollary 3.5). As a consequence of Corollary 3.5, we get the following result which may be viewed as a generalization of the Schur positivity [1] (see Corollary 3.6): Given two partitions $(\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p \geq 0)$