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Embedding in factorisable restriction monoids ☆

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ABSTRACT

Each restriction semigroup is proved to be embeddable in a factorisable restriction monoid, or, equivalently, in an almost factorisable restriction semigroup. It is also established that each restriction semigroup has a proper cover which is embeddable in a semidirect product of a semilattice by a group.

1. Introduction

Restriction semigroups are non-regular generalisations of inverse semigroups. They are semigroups equipped with two additional unary operations which satisfy certain identities. In particular, each inverse semigroup determines a restriction semigroup where

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the unary operations assign the idempotents aa^{-1} and $a^{-1}a$, respectively, to any element a. The class of restriction semigroups is just the variety of algebras generated by these restriction semigroups obtained from inverse semigroups, see [3]. Restriction semigroups (formerly called weakly E-ample semigroups) have arisen from a number of mathematical perspectives. For a historical overview of restriction semigroups and a detailed introduction to their fundamental properties the reader is referred to [5].

So far, a number of important results of the rich structure theory of inverse semigroups have been recast in the broader setting of restriction semigroups. The current paper is a contribution to this body of work. Our primary goal is to prove the following (see Theorem 3.10).

Main result *Any restriction semigroup is embeddable in a factorisable restriction monoid as a restriction semigroup, that is, in a way that respects both unary operations.*

Earlier works [7,8,11,12] have achieved embeddings of some or all restriction semigroups in members of wider classes that are inherently one-sided. The difficulty of our task is perhaps understood when we remark that it is undecidable whether a finite restriction semigroup embeds as a restriction semigroup into an inverse semigroup [6]. To this end we need to develop a proof strategy that is different in some crucial aspects from earlier ones.

In the theory of inverse semigroups, semidirect products of semilattices by groups play an important role. One of the reasons for this is that every inverse semigroup divides such a semidirect product (see [9] for the main results in this area and for references). In fact, something stronger is true, namely that every inverse semigroup can be embedded into an (idempotent separating) homomorphic image of such a semidirect product. One way to see this result is through extensions of partial isomorphisms of semilattices: an inverse semigroup determines partial isomorphisms of its semilattice of idempotents, and constructing the corresponding semidirect product in fact includes embedding this semilattice into a bigger one, and finding a group acting on the bigger semilattice such that the original partial isomorphisms are restrictions of the automorphisms determined by the group action.

The situation for restriction semigroups is similar, but also different in some crucial respects. In this case groups are replaced by monoids, which gives rise to several problems. The first result analogous to that formulated above for inverse semigroups was obtained by the third author in [12]: she has shown that every restriction semigroup can be embedded into a (projection separating) homomorphic image of a so-called W-product of a semilattice by a monoid, that is, into an almost left factorisable restriction semigroup (cf. [4]). An alternative proof, based on the idea of extending partial isomorphisms to injective endomorphisms, has been presented by Kudryavtseva in [8].

In a W-product in general, the monoid acts by injective endomorphisms on the semilattice, and this was the case in the results of [12] and [8] mentioned. In particular, if the monoid acts by automorphisms on the semilattice then the W-product becomes a semidirect product.