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Embedding properties of hereditarily just infinite profinite wreath products

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ABSTRACT

We study infinitely iterated wreath products of finite permutation groups w.r.t. product actions. In particular, we prove that, for every non-empty class of finite simple groups $\mathcal{X}$, there exists a finitely generated hereditarily just infinite profinite group W with composition factors in $\mathcal{X}$ such that any countably based profinite group with composition factors in $\mathcal{X}$ can be embedded into W. Additionally we investigate when infinitely iterated wreath products of finite simple groups w.r.t. product actions are co-Hopfian or non-co-Hopfian.

1. Introduction and main results

1.1. Introduction

A profinite group G is just infinite if G is infinite and every non-trivial closed normal subgroup $N \trianglelefteq_c G$ is open in G. While a complete classification of just infinite profinite groups is way out of reach, there is a natural interest in understanding as much about

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their structure as possible. It is known (e.g., see [7, Theorem 3]) that every just infinite profinite group either is a profinite branch group or contains an open subgroup isomorphic to the direct product of a finite number of copies of a hereditarily just infinite profinite group, where a profinite group G is called *hereditarily just infinite* if every open subgroup $H \leq_{\mathrm{o}} G$ is just infinite. While branch groups have been studied quite extensively (e.g., see [2]) comparatively little is known about hereditarily just infinite groups.

Well-known families of hereditarily just infinite profinite groups are supplied by compact open subgroups of simple algebraic groups over non-archimedean local fields, e.g., groups such as $\mathrm{SL}_n(\mathbb{Z}_p)$ or $\mathrm{SL}_n(\mathbb{F}_p[\![t]\!])$; see [8]. In addition there are some 'sporadic' non-linear examples, such as $\mathrm{Aut}(\mathbb{F}_p[\![t]\!])$ and certain subgroups thereof; see [3,1,5]. In [16, Theorem A], J. S. Wilson gave the first examples of hereditarily just infinite profinite groups that are not virtually pro-p for any prime p. They arise as certain iterated wreath products of non-abelian finite simple groups, and retrospectively the construction is very flexible. In [16,10,14] some embedding, generation and presentation properties of such groups have been established, but many of their features are not yet fully understood. In passing, we remark that A. Lucchini has used crown-based powers to manufacture further examples of hereditarily just infinite profinite groups; see [9]. Interesting new types of hereditarily just infinite pro-p groups were constructed by Ershov and Jaikin in [6].

In this paper we focus on hereditarily just infinite profinite groups that are obtained as inverse limits of iterated wreath products w.r.t. product actions. They arise as follows; see Section 2 for a more detailed description. Let $\mathcal{S} = (S_k)_{k \in \mathbb{N} \cup \{0\}}$, with $S_k \leq \mathrm{Sym}(\Omega_k)$, be a sequence of finite transitive permutation groups. The inverse limit

$$W^{\mathrm{pa}}(\mathcal{S}) = \varprojlim W_n^{\mathrm{pa}}$$

of the inverse system $W_0^{\mathrm{pa}} \twoheadleftarrow W_1^{\mathrm{pa}} \twoheadleftarrow \dots$ of finite iterated wreath products w.r.t. product actions

$$W_n^{\mathrm{pa}} = S_n \wr (S_{n-1} \wr (\cdots \wr S_0)) \leq \mathrm{Sym}(\widehat{\Omega}_n) \quad \text{for } \widehat{\Omega}_n = \Omega_n^{(\Omega_{n-1}^{(\cdots^{\Omega_0})})},$$

is called the *infinitely iterated wreath product of type $\mathcal{S}$ w.r.t. product actions.*

By [11, Theorem 6.2] and [10], every infinitely iterated wreath product w.r.t. product actions $W^{\mathrm{pa}}(\mathcal{S})$, based on a sequence $\mathcal{S}$ of finite non-abelian simple permutation groups, is a finitely generated hereditarily just infinite profinite group that is not virtually pro-p for any prime p.

1.2. Main results

The aim of this paper is to study embedding properties of infinitely iterated wreath products of finite non-abelian simple groups w.r.t. product actions. Specifically, we are interested in embeddings of countably based profinite groups with specified (topological)