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Lie bialgebras, fields of cohomological dimension at most 2 and Hilbert's Seventeenth Problem

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ABSTRACT

We investigate Lie bialgebra structures on simple Lie algebras of non-split type A. It turns out that there are several classes of such Lie bialgebra structures, and it is possible to classify some of them. The classification is obtained using Belavin–Drinfeld cohomology sets, which are introduced in the paper. Our description is particularly detailed over fields of cohomological dimension at most two, and is related to quaternion algebras and the Brauer group. We then extend the results to certain rational function fields over real closed fields via Pfister's theory of quadratic forms and his solution to Hilbert's Seventeenth Problem.

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1. Introduction

The study of quantum groups was initiated by Kulish and Reshetikhin in [11] and developed independently by Drinfeld [2] and Jimbo [8] in the 1980s. Over the past three decades, the area has seen major activity in various directions.

Quantum groups are deformations of universal enveloping algebras of Lie algebras. More specifically, if F is a field of characteristic zero, then by a *quantum group* we understand a topologically free Hopf algebra $U_\hbar$ over the ring $F[[\hbar]]$ of formal power series in F, such that, over F, the quotient $U_\hbar/\hbar U_\hbar$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ of some F-Lie algebra $\mathfrak{g}$. In [3] and [4], Etingof and Kazhdan constructed their quantization functors, thereby establishing an equivalence of categories that relates the problem of classifying quantum groups to that of classifying Lie bialgebras over $F[[\hbar]]$. If $\mathfrak{g}$ is finite-dimensional, the problem can largely be reduced to the classification of Lie bialgebra structures on the scalar extension $\mathfrak{g}_{F((\hbar))}$ of $\mathfrak{g}$ to the field $F((\hbar))$. This spurred the motivation to classify Lie bialgebras over fields of characteristic zero which are not algebraically closed.

Over algebraically closed fields, Lie bialgebra structures on simple Lie algebras have been classified by Belavin and Drinfeld [1]. Over non-closed fields, results have been obtained by Stolin and co-authors, upon introducing a cohomology theory known as *Belavin–Drinfeld cohomology*. This descent-type method resembles that of Galois cohomology, and has been applied to various split Lie algebras over fields which are not algebraically closed. The aim of this paper is to extend it to non-split Lie algebras. We investigate the situation for such algebras of type A.

More specifically, over a field F of characteristic zero with a quadratic field extension $K = F(\sqrt{d})$, we consider the Lie algebra $\mathfrak{su}(n, F, d)$ of all $A \in \mathfrak{sl}(n, K)$ satisfying $\overline{A}^T + A = 0$, where the conjugation $A \mapsto \overline{A}$ is induced by the non-trivial element of the Galois group $\mathrm{Gal}(K/F)$. We then ask when a Lie bialgebra structure on $\mathfrak{sl}(n, K)$ descends to $\mathfrak{su}(n, F, d)$, and study the behaviour of these structures. Over the algebraic closure $\overline{F}$, any such structure is a coboundary Lie bialgebra, gauge equivalent to the coboundary of λr_{BD} for some $\lambda \in \overline{F}$ and a non-skew symmetric r-matrix r_{BD} in the Belavin–Drinfeld classification. We prove that there are three possibilities for λ; namely, up to a scalar multiple in F, we have $\lambda = 1$, $\lambda = \sqrt{d}$ and $\lambda = \sqrt{d'}$ for some $d' \in K^* \setminus K^{*2}$. We will refer to these three types of Lie bialgebra structures as *basic*, *quadratic* and *twisted*, respectively. In the quadratic and twisted case, we show that such Lie bialgebra structures exist only if r_{BD} is essentially of Drinfeld–Jimbo type. Our investigation is particularly detailed in the quadratic case, where the Drinfeld double of the Lie bialgebra is $\mathfrak{sl}(n, K)$ itself. There we achieve a classification of these Lie bialgebra structures for r-matrices of Drinfeld–Jimbo type, over fields of cohomological dimension at most 2, as well as over function fields in at most 2 indeterminates over real-closed fields.

The paper is organized as follows. In Section 2 we give the necessary preliminaries and then focus on preparing the setting for the definition and characterization of the necessary cohomology theory in the case where the Lie bialgebra is of quadratic Drinfeld–Jimbo