Journal of Functional Analysis 273 (2017) 875–916

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

# Semiclassical limit of Liouville Field Theory

Hubert Lacoin [a], Rémi Rhodes [a,*,1], Vincent Vargas [b,1]

[a] Université Paris-Dauphine, Ceremade, F-75016 Paris, France
[b] Ecole Normale Supérieure, DMA, 45 rue d'Ulm, 75005 Paris, France

A R T I C L E I N F O

*Article history:*
Received 17 June 2014
Accepted 25 April 2017
Available online 3 May 2017
Communicated by L. Gross

*Keywords:*
Gaussian multiplicative chaos
Semiclassical limit
Large deviation principle
Liouville field theory

A B S T R A C T

Liouville Field Theory (LFT for short) is a two dimensional model of random surfaces, which is for instance involved in $2d$ string theory or in the description of the fluctuations of metrics in $2d$ Liouville quantum gravity. This is a probabilistic model that consists in weighting the classical Free Field action with an interaction term given by the exponential of a Gaussian multiplicative chaos. The main input of our work is the study of the semiclassical limit of the theory, which is a prescribed asymptotic regime of LFT of interest in physics literature (see [35] and references therein). We derive exact formulas for the Laplace transform of the Liouville field in the case of flat metric on the unit disk with Dirichlet boundary conditions. As a consequence, we prove that the Liouville field concentrates on the solution of the classical Liouville equation with explicit negative scalar curvature. We also characterize the leading fluctuations, which are Gaussian and massive, and establish a large deviation principle. Though considered as an ansatz in the whole physics literature, it seems that it is the first rigorous probabilistic derivation of the semiclassical limit of LFT. On the other hand, we carry out the same analysis when we further weight the Liouville

* Corresponding author.
*E-mail address:* remi.rhodes@u-pem.fr (R. Rhodes).

1 Partially supported by grant ANR-11-JCJC CHAMU.

http://dx.doi.org/10.1016/j.jfa.2017.04.012

action with heavy matter operators. This procedure appears when computing the $n$-points correlation functions of LFT.


## 1. Introduction

One of the aims of this paper is to initiate the study of Laplace asymptotics and large deviation principles in the realm of Liouville field theory.

To begin with, let us mention that there exists a considerable literature devoted to Laplace asymptotic expansions and large deviation principles for the canonical random paths: the Brownian motion in $\mathbb{R}^d$. To make things simple, the aim of these studies is to investigate the asymptotic behavior as $\gamma \to 0$ of

$$\mathbb{E}[G(\gamma B)e^{-\gamma^{-2}F(\gamma B)}] \tag{1.1}$$

where $B$ is a Brownian motion and $F$, $G$ are general functionals. Schilder's pioneering work [49] (see also [43]) treated the full asymptotic expansion in the case of Wiener integrals. This was then extended by Freidlin and Wentzell [28] to Itô diffusions. Similar results were obtained for conditioned Brownian paths (such as the Brownian bridge) by Davies and Truman [14–17]. Ellis and Rosen [25–27] also developed further Laplace asymptotic expansions for Gaussian functional integrals. Then Azencott and Doss [3] used asymptotic expansions to study the semiclassical limit of the Schrödinger equation (see also Azencott [1,2]). These works initiated a long series (see for instance [7–10]) and it is beyond the scope of this paper to review the whole literature until nowadays.

There is an important conceptual difference between canonical random paths and canonical random surfaces. Whereas Brownian motion and its variants are rather nicely behaved (Hölder continuous), the canonical two dimensional random surface, i.e. the Gaussian Free Field (GFF), is much wilder: it cannot be defined pointwise for instance and must be understood as a random distribution (like e.g. the local time of the one dimensional Brownian motion, or its derivative). As a consequence, many nonlinear functionals defined solely on the space of continuous functions must be defined via renormalization techniques when applied to the GFF: see the book of Simon [50] for instance. In this paper, we consider probably the most natural framework of weighted random surfaces: the $2d$-Liouville Field Theory (LFT). LFT is ruled by the Liouville action

$$S_L(\varphi) = \frac{1}{4\pi}\int_D \left[|\partial^{\hat{g}}\varphi|^2_{\hat{g}} + QR_{\hat{g}}\varphi + 4\pi\mu e^{\gamma\varphi}\right]\lambda_{\hat{g}}(\mathrm{d}x) \tag{1.2}$$

in the background metric $\hat{g}$ ($\partial^{\hat{g}}$, $R_{\hat{g}}$ and $\lambda_{\hat{g}}$ stand for the gradient, curvature and volume form of the metric $\hat{g}$) with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\gamma \in ]0, 2]$ and $\mu > 0$. This is a model describing