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Stationary points of nonlinear plate theories ☆

Peter Hornung

Fachbereich Mathematik, TU Dresden, Germany

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We consider bending theories for thin elastic films obtained by endowing a bounded domain $S \subset \mathbb{R}^2$ with a Riemannian metric g. The associated elastic energy is given by a nonlinear isometry-constrained bending energy functional, which is a natural generalisation of Kirchhoff's plate functional to metrics with possibly nonzero Gauss curvature. We introduce and discuss a natural notion of stationarity for such functionals.

We then show that all rotationally symmetric immersions of the unit disk are stationary in that sense, and we give examples of smooth metrics g leading to functionals with infinitely many stationary points. Finally, we implement our general approach in the case when g has positive Gauss curvature.

1. Introduction

Thin growing tissues in biology or elastic sheets obtained by controlled polymerisation (cf. [27]) display bending patterns even if no external forces are applied. Such situations are characterised by the fact that the reference configuration is intrinsically strained and is therefore not stress-free.

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E-mail address: peter.hornung@tu-dresden.de.

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Such questions have been studied in [7,14,27,10,9,8] and elsewhere. Two scenarios can arise: either there is a way of fully relaxing the internal stresses by deforming the sheet (with finite bending energy), or this is not possible. In this article we are interested in the former case.

The reference configuration of the sheet is modelled by a bounded domain $S \subset \mathbb{R}^2$ and the internal strains are modelled by a Riemannian 'target' metric g on S. The fact that there is a way of fully relaxing the internal stresses by deforming the sheet with finite bending energy means precisely that the set

$$W_g^{2,2}(S) = \left\{u \in W^{2,2}(S, \mathbb{R}^3) : (\nabla u)^T(\nabla u) = g \text{ almost everywhere in } S\right\}$$

of $W^{2,2}$ isometric immersions of (S, g) into $\mathbb{R}^3$ is nonempty. If this is the case, then the asymptotic behaviour of the three dimensional elastic energy of the leaf with small thickness is captured by the natural generalisation of Kirchhoff's nonlinear bending theory of plates which we introduce next. Firstly, recall that for any regular immersion $u : S \to \mathbb{R}^3$ the Willmore functional (cf. e.g. [52,45]) is given by

$$W(u) = \frac{1}{4}\int_S H^2 d\mu_g + \int_{\partial S} \kappa_g d\mu_{g_\partial}, \tag{1}$$

where κ_g is the geodesic curvature of ∂S and g is the metric induced by u, the induced area measure on S is μ_g and the induced boundary measure on ∂S is μ_{g_∂}. The Willmore functional has been studied extensively in the literature, cf. [54,47,41] and the references cited therein. Its analytical properties have been systematically studied in [32,2,33,43]. More recently, there has been growing interest in constrained versions of the Willmore functional. The typical constraints include prescribed conformal class (cf. [3,34,44]) or fixed area and enclosed volume, cf. [18,46].

The relevant case for thin film elasticity is the restriction of the Willmore functional (1) to *isometric* immersions of the given Riemannian manifold (S, g) into $\mathbb{R}^3$. More precisely, from now on $S \subset \mathbb{R}^2$ will denote a bounded simply connected domain with smooth boundary, and $g : \overline{S} \to \mathbb{R}^{2\times 2}$ will be a given smooth Riemannian metric on $\overline{S}$. We will study the restriction of the Willmore functional to the class $W_g^{2,2}(S)$. That is, we will study the generalised Kirchhoff plate functionals

$$\widetilde{\mathcal{W}}_g(u) = \begin{cases} W(u) & \text{if } u \in W_g^{2,2}(S) \\ +\infty & \text{otherwise.} \end{cases}$$

In addition to their key role in the modelling of thin films in nonlinear elasticity, these functionals are also natural from a geometric viewpoint. It is a key feature of thin films in nonlinear elasticity that they undergo large deformations with low energy. In contrast to constrained von-Kármán theories, the above functionals admit such large deformations. In fact, they arise naturally as (rigorous) asymptotic theories from fully nonlinear