Journal of Functional Analysis 273 (2017) 984–1019

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Tests for complete K-spectral sets ☆

Michael A. Dritschel [a,*], Daniel Estévez [b], Dmitry Yakubovich [b,c]

[a] *School of Mathematics and Statistics, Herschel Building, University of Newcastle, Newcastle upon Tyne, NE1 7RU, UK*
[b] *Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco 28049, Madrid, Spain*
[c] *Instituto de Ciencias Matemáticas (CSIC–UAM–UC3M–UCM), Spain*

A R T I C L E I N F O

Article history:
Received 30 October 2015
Accepted 16 March 2017
Available online 28 March 2017
Communicated by D. Voiculescu

MSC:
primary 47A25
secondary 30H50, 46J15, 47A12, 47A20

Keywords:
Spectral sets and dilations
Separation of singularities
Algebra generation
Schur–Agler algebra

A B S T R A C T

Let Φ be a family of functions analytic in some neighborhood of a complex domain Ω, and let T be a Hilbert space operator whose spectrum is contained in $\overline{\Omega}$. Our typical result shows that under some extra conditions, if the closed unit disc is complete K'-spectral for $\varphi(T)$ for every $\varphi \in \Phi$, then $\overline{\Omega}$ is complete K-spectral for T for some constant K. In particular, we prove that under a geometric transversality condition, the intersection of finitely many K'-spectral sets for T is again K-spectral for some $K \geq K'$. These theorems generalize and complement results by Mascioni, Stessin, Stampfli, Badea–Beckermann–Crouzeix and others. We also extend to non-convex domains a result by Putinar and Sandberg on the existence of a skew dilation of T to a normal operator with spectrum in $\partial\Omega$. As a key tool, we use the results from our previous paper [11] on traces of analytic uniform algebras.

☆ The three authors have been supported by Project MTM2015-66157-C2-1-P of the Ministry of Economy and Competitiveness of Spain. The second author acknowledges a post-graduate grant of the Mathematics Department of the Universidad Autónoma de Madrid. The third author has also been supported by the ICMAT Severo Ochoa project SEV-2015-0554 of the Ministry of Economy and Competitiveness of Spain and the European Regional Development Fund (FEDER).
* Corresponding author.
E-mail addresses: michael.dritschel@newcastle.ac.uk (M.A. Dritschel), daniel.estevez@uam.es (D. Estévez), dmitry.yakubovich@uam.es (D. Yakubovich).

http://dx.doi.org/10.1016/j.jfa.2017.03.008
0022-1236/© 2017 Published by Elsevier Inc.

1. Introduction

Let T be an operator on a Hilbert space H and Ω a bounded subset of $\mathbb{C}$ containing the spectrum $\sigma(T)$. We recall that, given a constant $K \geq 1$, the closure $\overline{\Omega}$ of Ω is said to be *a complete K-spectral set for T* if the matrix von Neumann inequality

$$\|p(T)\|_{\mathcal{B}(H\otimes\mathbb{C}^s)} \leq K \max_{z\in\overline{\Omega}} \|p(z)\|_{\mathcal{B}(\mathbb{C}^s)} \tag{1}$$

holds for any square $s \times s$ rational matrix function $p(z)$ of any size s and with poles off of $\overline{\Omega}$; here $\mathcal{B}(H)$ denotes the space of linear operators on H. The set $\overline{\Omega}$ is called a *K-spectral set for T* if (1) holds for $s = 1$. By a well-known theorem of Arveson [3], $\overline{\Omega}$ is a complete K-spectral set for T for some $K \geq 1$ if and only if T is similar to an operator, which has a normal dilation N with $\sigma(N) \subset \partial\Omega$; the importance of complete K-spectral sets is due to this result.

We denote by $\widehat{\mathbb{C}}$ the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By a *Jordan domain* in $\widehat{\mathbb{C}}$ we mean an open domain $\Omega \subset \widehat{\mathbb{C}}$ whose boundary is a Jordan curve. A Jordan domain (or Jordan domain in $\mathbb{C}$) is just a bounded Jordan domain in $\widehat{\mathbb{C}}$. A curve $\Gamma \subset \mathbb{C}$ is called *Ahlfors regular* if $|B(z,\varepsilon) \cap \Gamma| \leq C\varepsilon$, for every $\varepsilon > 0$ and every $z \in \Gamma$, where C is a constant independent of ε and z. Here $|\cdot|$ denotes the arc-length measure and $B(z,\varepsilon)$ is the open disk of radius ε and center z.

By a *circular sector with vertex z_0* we mean a set in $\mathbb{C}$ of the form

$$\{z \in \mathbb{C} : 0 < |z - z_0| < r,\ \alpha < \arg z < \beta\},$$

where $r > 0$ and $\alpha, \beta \in \mathbb{R}$, $0 < \beta - \alpha < 2\pi$. The aperture of such a circular sector is the number $\beta - \alpha$.

If $\Omega_1, \Omega_2 \subset \widehat{\mathbb{C}}$ are two open sets, and $\infty \neq z_0 \in \partial\Omega_1 \cap \partial\Omega_2$ is a point in the intersection of their boundaries, we say that the boundaries of Ω_1 and Ω_2 *intersect transversally at* z_0 if one can find five pairwise disjoint circular sectors $S_0, S_1^l, S_1^r, S_2^l, S_2^r$ with vertex z_0, having the same aperture, and such that the following conditions are satisfied:

- S_0 does not intersect $\overline{\Omega}_1 \cup \overline{\Omega}_2$.
- $B(z_0,\varepsilon) \cap \partial\Omega_j \subset S_j^l \cup S_j^r \cup \{z_0\}$ for $j = 1, 2$ and some $\varepsilon > 0$.
- For every $\delta > 0$, $B(z_0,\delta) \cap \Omega_1 \cap \Omega_2$ is not empty.

In the case when $\infty \in \partial\Omega_1 \cap \partial\Omega_2$, we say that the boundaries of Ω_1 and Ω_2 intersect transversally at ∞ if the boundaries of $\psi(\Omega_1)$ and $\psi(\Omega_2)$ intersect transversally at 0, where $\psi(z) = 1/z$. We say that the boundaries of Ω_1 and Ω_2 intersect transversally if they intersect transversally at every point of $\partial\Omega_1 \cap \partial\Omega_2$. Note that the third condition in the definition of a transversal intersection implies that $\overline{\Omega_1 \cap \Omega_2} = \overline{\Omega}_1 \cap \overline{\Omega}_2$.

By an analytic arc in $\mathbb{C}$ we mean an image of the interval $[0, 1]$ under a function, analytic in its neighborhood. A piecewise analytic curve will mean a curve which can be subdivided into finitely many analytic arcs.