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COMPOSITIONS OF CONDITIONAL EXPECTATIONS, AMEMIYA-ANDÔ CONJECTURE AND PARADOXES OF THERMODYNAMICS

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Abstract. We investigate properties of compositions of conditional expectations on a non-atomic probability space $(\Omega, \mathcal{F}, \mu)$. Let $1 \leq p < \infty$ and $X, Y \in L^p(\Omega, \mathcal{F}, \mu)$. If for any convex $f : \mathbb{R} \to \mathbb{R}$ we have $Ef(X) \geq Ef(Y)$, then for each $\varepsilon > 0$ there exist conditional expectations P_1, P_2, P_3 and $E_1, \ldots, E_n \in \{P_1, P_2, P_3\}$ such that $\|Y - E_n \ldots E_1 X\|_p < \varepsilon$. This theorem seems particularly surprising if one simplifies the assumption by taking X and Y to have the same distribution since, intuitively, it seems to contradict the second law of thermodynamics. We use this theorem to build the following counterexample for the Amemiya-Andô conjecture: There exist $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, conditional expectations $P_1, \ldots, P_5$ on $(\mathbb{R}, Borel(\mathbb{R}), \lambda)$ and $E_1, E_2, \ldots \in \{P_1, \ldots, P_5\}$ such that the sequence of iterations $(E_n \ldots E_2 E_1 f)$ diverges in $L^2(\mathbb{R})$.

1. Introduction and main results

This article is motivated by a recent solution of an old problem stated by I. Amemiya and T. Andô in 1965 in [1]. Let H be a real Hilbert space, $x \in H$ and let $P_1, \ldots, P_k$ be orthogonal projections in H. We consider any sequence $E_1, E_2, \ldots \in \{P_1, \ldots, P_k\}$. Is it true that the sequence of iterations $(E_n \ldots E_2 E_1 x)$ is convergent? I. Amemiya and T. Andô showed that the sequence of iterations is weakly convergent and they conjectured that it is also convergent in norm of H. In recent years A. Paszkiewicz [21, 22], E. Kopecká and V. Müller [13] constructed counterexamples for the Amemiya-Andô conjecture.

Our intention was to provide a new counterexample such that $H = L^2(\Omega, \mathcal{F}, \mu)$ for a probability space $(\Omega, \mathcal{F}, \mu)$ and $P_1, \ldots, P_k$ are conditional expectations. The idea is connected with our recent result [12]: If a pair (P, Q) of orthogonal projections on a separable Hilbert space satisfies $\operatorname{rank}(P \wedge Q) = \infty$, then (P, Q) is unitarily equivalent to a pair of conditional expectations on L^2 for some probability space. This result encouraged us to transfer the problem of the Amemiya-Andô conjecture from the world of Hilbert spaces and orthogonal projections to the world of L^2 spaces and conditional expectations. The problem for probability spaces is still open. Instead, we prove the following result (see Theorem 20 in Section 5): There exist five conditional expectations P_1, P_2, P_3, P_4, P_5 on the measure space $(\mathbb{R}, Borel(\mathbb{R}), \lambda)$, an element $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and a sequence $E_1, E_2, \ldots \in \{P_1, P_2, P_3, P_4, P_5\}$ such that the sequence $(E_n \cdots E_2 E_1 f)$ diverges in $L^2(\mathbb{R})$. The measure space $(\mathbb{R}, Borel(\mathbb{R}), \lambda)$ is not a probability space, therefore it is necessary to extend the definition of conditional expectation (see Definition 19).

Obtaining this theorem required to develop some results concerning compositions of conditional expectations on a probability space. These results (which are, in our opinion, much more interesting than the above counterexample) are presented in Sections 2, 3 and 4.

Our study of compositions of conditional expectations is started in Section 2, where we consider pairs (X, Y) of random variables defined on a non-atomic probability space, sharing the same distribution and

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