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Anderson localization for one-frequency quasi-periodic block Jacobi operators

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We consider a one-frequency, quasi-periodic, block Jacobi operator, whose blocks are generic matrix-valued analytic functions. We establish Anderson localization for this type of operator under the assumption that the coupling constant is large enough but independent of the frequency. This generalizes a result of J. Bourgain and S. Jitomirskaya on localization for band lattice, quasi-periodic Schrödinger operators.

1. Introduction and statement

An integer lattice quasi-periodic Schrödinger operator is an operator $H_\lambda(x)$ on $l^2(\mathbb{Z}) \ni \psi = \{\psi_n\}_{n\in\mathbb{Z}}$, defined by

$$[H_\lambda(x)\,\psi]_n := -(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \lambda\, f(x + n\omega)\,\psi_n\,,$$

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where $\lambda \neq 0$ is a coupling constant, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a phase parameter that introduces some randomness into the system, $f\colon \mathbb{T} \to \mathbb{R}$ is a (bounded) potential function, and $\omega \in \mathbb{T}$ is a fixed irrational frequency.

Note that $H_\lambda(x)$ is a bounded, self-adjoint operator. Moreover, due to the ergodicity of the system, the spectral properties of the family of operators $\{H_\lambda(x)\colon x \in \mathbb{T}\}$ are independent of x almost surely.

In this paper we study a more general Schrödinger-like operator on a *band* integer lattice (which in some sense may be regarded as an approximation of a higher dimensional lattice). Before we define such operators, let us introduce some notations and terminology.

All throughout, if $m \in \mathbb{N}$ and if $M\colon \mathbb{T} \to \mathrm{Mat}_m(\mathbb{R})$ is any matrix-valued function, we denote by $M^\top(x)$ the transpose of $M(x)$.

We say that $M(x)$ has no constant eigenvalues if for any $w \in \mathbb{C}$, we have $\det[M(x) - w\,I] \not\equiv 0$ as a function of x.

Furthermore, given a frequency $\omega \in \mathbb{T}$, for all $n \in \mathbb{Z}$ we denote by $M_n(x)$ the quasi-periodic matrix-valued function

$$M_n(x) := M(x + n\omega).$$

All such matrix-valued functions M will be assumed real *analytic* (meaning that their entries are real analytic functions).

Let us then denote by $C_r^\omega(\mathbb{T}, \mathrm{Mat}_m(\mathbb{R}))$ the space of all analytic functions $M\colon \mathbb{T} \to \mathrm{Mat}_m(\mathbb{R})$ having a holomorphic, continuous up to the boundary extension[1] to $\mathcal{A}_r := \{z \in \mathbb{C}\colon 1 - r < |z| < 1 + r\}$, the annulus of width $2r$ around the torus $\mathbb{T}$. We endow this space with the uniform norm $\|M\|_r := \sup_{z \in \mathcal{A}_r} \|M(z)\|$.

Let $l \in \mathbb{N}$ be the width of the band lattice, fix an irrational frequency $\omega \in \mathbb{T}$ and let $W, R, F \in C_r^\omega(\mathbb{T}, \mathrm{Mat}_l(\mathbb{R}))$. Assume that for all phases $x \in \mathbb{T}$, $R(x)$ and $F(x) \in \mathrm{Sym}_l(\mathbb{R})$, i.e they are symmetric matrices.

A *quasi-periodic block Jacobi operator* is an operator $H = H_\lambda(x)$ acting on $l^2(\mathbb{Z} \times \{1, \dots, l\}, \mathbb{R}) \simeq l^2(\mathbb{Z}, \mathbb{R}^l)$ by

$$[H_\lambda(x)\,\vec{\psi}]_n := -(W_{n+1}(x)\,\vec{\psi}_{n+1} + W_n^\top(x)\,\vec{\psi}_{n-1} + R_n(x)\,\vec{\psi}_n) + \lambda\,F_n(x)\,\vec{\psi}_n, \qquad (1.1)$$

where $\vec{\psi} = \{\vec{\psi}_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{R}^l)$ is any state, and as before, $x \in \mathbb{T}$ is a phase and $\lambda \neq 0$ is a coupling constant.

This model contains all quasi-periodic, finite range hopping Schrödinger operators on integer or band integer lattices. The hopping term is given by the "weighted" Laplacian:

$$[\Delta_W(x)\,\vec{\psi}]_n := W_{n+1}(x)\,\vec{\psi}_{n+1} + W_n^\top(x)\,\vec{\psi}_{n-1} + R_n(x)\,\vec{\psi}_n\,,$$

[1] We warn the reader that we identify the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (an additive group) with the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ (a multiplicative group) via the map $x + \mathbb{Z} \mapsto e(x) := e^{2\pi i x}$, but we maintain the additive notation, e.g. we write $x + \omega$ instead of $e(x)e(\omega)$.