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A nonlinear time compactness result and applications to discretization of degenerate parabolic–elliptic PDEs

Boris Andreianov [a], Clément Cancès [b], Ayman Moussa [c,d,*]

[a] *Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Université de Tours, 37200 Tours, France*
[b] *Team Rapsodi, Inria Lille – Nord Europe, 59650, Villeneuve d'Ascq, France*
[c] *Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 75005, Paris, France*
[d] *CNRS UMR 7598, Laboratoire Jacques-Louis Lions, 75005, Paris, France*

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ABSTRACT

We propose a discrete functional analysis result suitable for proving compactness in the framework of fully discrete approximations of strongly degenerate parabolic problems. It is based on the original exploitation of a result related to compensated compactness rather than on a classical estimate on the space and time translates in the spirit of Simon [49]. Our approach allows to handle various numerical discretizations both in the space variables and in the time variable. In particular, we can cope quite easily with variable time steps and with multistep time differentiation methods like, e.g., the backward differentiation formula of order 2 (BDF2) scheme. We illustrate our approach by proving the convergence of a two-point flux Finite Volume in space and BDF2 in time approximation of the porous medium equation.

* Corresponding author.
E-mail addresses: boris.andreianov@lmpt.univ-tours.fr (B. Andreianov), clement.cances@inria.fr (C. Cancès), moussa@ljll.math.upmc.fr (A. Moussa).

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1. Introduction

There exists a large variety of numerical strategies for discretization of evolution PDEs. Proofs of convergence of many different numerical schemes often take the following standard itinerary (see, e.g., [29]). Given a PDE, discrete equations of the scheme are rewritten under a form reminiscent of the weak formulation of the continuous problem; stability estimates are obtained, which ensure bounds in appropriate (possibly discretized) functional spaces; eventually, sufficiently strong compactness arguments permit to pass to the limit in the discrete weak formulation.

In many applications, including degenerate parabolic equations of various kinds, the question of strong compactness in L^p spaces of sequences of approximate solutions is a cornerstone of such proofs. While "space compactness" is usually obtained by suitable *a priori* estimates of the discrete gradients involved in the equation, "time compactness" is often obtained by explicitly estimating L^2 or L^1 time translates in the spirit of Alt and Luckhaus [2]. This step is equation-dependent, and has to be reproduced for each problem. We refer to [29,31] for the main ingredients of this already classical argument used in a number of subsequent works, to [6,4] for some refinements, and to [16, §3] for a shortened version of the argument.

The same question of time compactness often arises in existence analysis for PDEs, in the continuous framework. Along with the technique of [2], there exist several ready-to-use results that yield space-time precompactness of a sequence of (approximate) solutions $(u_n)_n$. They are based on the two following ingredients:

(A) estimates in sufficiently narrow Bochner spaces ensuring uniform in n bounds on space translates of the sequence $(u_n)_n$;

(B) some very weak (in the space variable) estimates on the sequence $(\partial_t u_n)_n$.

Then, different arguments permit to derive from (A) and (B) uniform in n estimates of time translates of $(u_n)_n$ and conclude that $(u_n)_n$ is compact in the appropriate space (e.g., as a consequence of the Fréchet–Kolmogorov compactness criterion for L^p spaces). This kind of result, in the abstract linear setting, is often called Aubin–Lions–Simon lemma [11,40,49], but the version we are interested in is also related to the early nonlinear version of the argument due to Dubinskii [26] (see also recent references [13,21,20]) and to the more recent formulation of Maître [41]. Another related argument of nonlinear kind is due to Kruzhkov [39]. Further improvements were obtained by Amann in [3] for a refined scale of spaces (including Besov spaces for instance), and broached by Roubíček in a rather general setting, see [47]. One observes that several closely related results co-exist, but the precise assumptions and conclusions of there results differ. Therefore, one can see the combination of properties $(A)\&(B)$ as a "time compactness principle", which can be made precise upon choosing a suitable functional framework or a suitable form of the estimates (A) and (B) (as one illustration, we refer to Emmrich and Thalhammer [27] where (B) is formulated as a fractional time derivative estimate).