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A non-compactness result on the fractional Yamabe problem in large dimensions

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ABSTRACT

Let (X^{n+1}, g^+) be an $(n+1)$-dimensional asymptotically hyperbolic manifold with conformal infinity $(M^n, [\hat{h}])$. The fractional Yamabe problem addresses to solve

$$P^\gamma[g^+, \hat{h}](u) = cu^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0 \quad \text{on } M$$

where $c \in \mathbb{R}$ and $P^\gamma[g^+, \hat{h}]$ is the fractional conformal Laplacian whose principal symbol is the Laplace–Beltrami operator $(-\Delta)^\gamma$ on M. In this paper, we construct a metric on the half space $X = \mathbb{R}^{n+1}_+$, which is conformally equivalent to the unit ball, for which the solution set of the fractional Yamabe equation is non-compact provided that $n \geq 24$ for $\gamma \in (0, \gamma^*)$ and $n \geq 25$ for $\gamma \in [\gamma^*, 1)$ where $\gamma^* \in (0, 1)$ is a certain transition exponent. The value of γ^* turns out to be approximately 0.940197.

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Contents

1. Introduction

Given $n \in \mathbb{N}$, let (X^{n+1}, g^+) be an $(n+1)$-dimensional asymptotically hyperbolic manifold with a conformal infinity $(M^n, [\hat{h}])$. In [43], Graham and Zworski introduced the fractional conformal Laplacian $P_{\hat{h}}^\gamma = P^\gamma[g^+, \hat{h}]$ for $\gamma \in (0, n/2)$ whose principal symbol is given as the Laplace–Beltrami operator $(-\Delta)^\gamma$ on M and which obeys the conformal covariance property:

$$P^\gamma \left[g^+, w^{\frac{4}{n-2\gamma}} \hat{h}\right] = w^{-\frac{n+2\gamma}{n-2\gamma}} P^\gamma \left[g^+, \hat{h}\right] (w\cdot) \tag{1.1}$$

holds for any positive function w on M (refer also to [67,52,20,41]). If we denote by $Q_{\hat{h}}^\gamma = P_{\hat{h}}^\gamma(1)$ the associated fractional scalar curvature and further assume that (X, g^+) is a Poincaré–Einstein manifold, then $P_{\hat{h}}^1$ and $Q_{\hat{h}}^1$ become the conformal Laplacian and the scalar curvature (up to constant multiples)

$$P_{\hat{h}}^1 = -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)} R_{\hat{h}}, \quad Q_{\hat{h}}^1 = \frac{n-2}{4(n-1)} R_{\hat{h}}$$