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Non-Asplund Banach spaces and operators

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Let W and Z be Banach spaces such that Z is separable and let $R : W \longrightarrow Z$ be a (continuous, linear) operator. We study consequences of the adjoint operator R^* having non-separable range. From our main technical result we obtain applications to the theory of basic sequences and the existence of universal operators for various classes of operators between Banach spaces. We also obtain an operator-theoretic characterisation of separable Banach spaces with non-separable dual.

1. Introduction

The 1970s saw significant progress made on the understanding of the Radon–Nikodým property in Banach spaces. Amongst the main achievements in this area of investigation was the proof that a dual Banach space X^* has the Radon–Nikodým property if and only if every separable subspace of X has separable dual. We refer the reader to the book [6] for references and a proof of this result, but mention here in particular that it was C. Stegall

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who in [32] provided the proof of the fact that if X^* has the Radon–Nikodým property then every separable subspace of X has separable dual. The main technical result in Stegall's paper, Corollary 1 of [32], provides structural consequences for a separable Banach space X and its dual X^* in the case that X^* is non-separable. The main technical result of the current paper, Theorem 3.1 below, serves a similar purpose in the study of separable Banach spaces with non-separable dual. Our approach, which in some ways is fundamentally similar to Stegall's, uses techniques developed recently in [3] to study the Szlenk index. We obtain new proofs of old results and a number of new results concerning Banach spaces and operators with non-separable dual.

One of the primary applications of Theorem 3.1 of the current paper is to the theory of basic sequences in Banach spaces. Our work to this end is based on the classical method of Mazur for producing subspaces with a basis and the more recent method of Johnson and Rosenthal [21] for producing quotients with a basis.

The other main application of our techniques is to the problem of finding universal elements for certain subclasses of the class $\mathscr{L}$ of all (bounded, linear) operators between Banach spaces. For operators $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(W,Z)$, where W, X, Y and Z are Banach spaces, we say that S *factors through* T (or, equivalently, that T *factors* S) if there exist $U \in \mathscr{L}(W,X)$ and $V \in \mathscr{L}(Y,Z)$ such that $VTU = S$. With this terminology, for a given subclass $\mathscr{C}$ of $\mathscr{L}$ we say that an operator $\Upsilon \in \mathscr{C}$ is *universal for* $\mathscr{C}$ if Υ factors through every element of $\mathscr{C}$. Typically $\mathscr{C}$ will be the complement $\complement\mathscr{I}$ of an operator ideal $\mathscr{I}$ in the sense of Pietsch [27] (that is, $\complement\mathscr{I}$ consists of all elements of $\mathscr{L}$ that do not belong to $\mathscr{I}$), or perhaps the restriction $\mathscr{J} \cap \complement\mathscr{I}$ of $\complement\mathscr{I}$ to a large subclass $\mathscr{J}$ of $\mathscr{L}$; e.g., $\mathscr{J}$ might denote a large operator ideal or the class of all operators having a specified domain or codomain. One may think of a universal element of the class $\mathscr{C}$ as a minimal element of $\mathscr{C}$ that is 'fixed' or 'preserved' by each element of $\mathscr{C}$.

The notion of universality for a class of operators goes back to the work of Lindenstrauss and Pełczyński, who showed in [22] that the summation operator $\Sigma : (a_n)_{n=1}^{\infty} \mapsto (\sum_{i=1}^{n} a_i)_{n=1}^{\infty}$ from ℓ_1 to ℓ_∞ is universal for the class of non-weakly compact operators. Soon after, W.B. Johnson [20] showed that the formal identity operator from ℓ_1 to ℓ_∞ is universal for the class of non-compact operators. This result of Johnson has been applied in the study of information-based complexity by Hinrichs, Novak and Woźniakowski in [17].

The universality result of primary importance for us is due to Stegall and establishes the existence of a universal non-Asplund operator. The Asplund operators have several equivalent definitions in the literature; in the current paper we say that an operator $T : X \longrightarrow Y$ is *Asplund* if $T|_Z \in \mathscr{X}^*$ for any separable subspace $Z \subseteq X$, where $\mathscr{X}^*$ denotes the class of operators whose adjoint has separable range. We refer the reader to Stegall's paper [33] for further properties and characterisations of Asplund operators.

Stegall's universal non-Asplund operator is defined in terms of the Haar system $(h_m)_{m=0}^{\infty} \subseteq C(\{0,1\}^\omega)$, where each factor $\{0,1\}$ is discrete and $\{0,1\}^\omega$ is equipped with its compact Hausdorff product topology. The Haar system is a monotone basis for $C(\{0,1\}^\omega)$ and may be defined as follows. Let $\mathcal{D}$ denote the set $\bigcup_{n<\omega}\{0,1\}^n$, the set